Combinatorics of Cohomology Rings of the Peterson Variety: Transpositions

Swan Klein, Connor Mooney
Advised by: Rebecca Goldin, Quincy Frias

George Mason University, Mason Experimental Geometry Lab

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Introduction and Motivation

- Let X and Y be two rings with additional $\mathbb{C}[t]$-module structure.
- X has module basis $\{\sigma_w | w \in S_n\}$ where S_n is the permutation group on n symbols.
- Y has module basis $\{p_A | A \subseteq \{1, 2, \cdots, n-1\}\}$.
- We have a surjective ring map $\iota^* : X \to Y$.
- Let $v_A = \Pi_{j \in A} s_j$ ordered with lower js to the left, where $s_j = (j, j+1)$. Then
$$p_A = \iota^* \sigma_{v_A}.$$
- Question: What is $\iota^* \sigma_w$ in terms of the $\{p_A\}$ for $w \neq v_A$?

Transposition Conjecture

Let (i,j) be the transposition of i,j, where $i<j$. Let $m \coloneqq j-i$. Then

$$\iota^*(\sigma_{(i,j)}) = \sum_{k=0}^{m-1}\sum_{h=0}^{k} h!\binom{k}{h}^2\binom{m-1}{k}^2 t^h p_{[1+i+k-m,j+k-h-1]}$$

excluding any terms where $1+i-k-m<1$ or $j+k-h\geq n$, where $[a,b]=\{a,a+1,\cdots b-1,b\}$.

Simplified Transposition Conjecture

Let $(1, j)$ be the transposition of $1, j$, where $j > 1$. Then

$$\iota^*(\sigma_{(1,j)}) = \sum_{h=0}^{j-2} h! \binom{j-2}{h}^2 t^h p_{[2j-h-3]},$$

where $[2j - h - 3] = \{1, 2, \cdots 2j - h - 3\}$.

Reduced words for $(1,j)$

The shortest possible strings of $s_i = (i, i+1)$ that multiply out to $(1,j)$. They have length $2j-3$. They are related by commutation (i.e. $s_1 s_3 = s_3 s_1$) and braid moves ($s_1 s_2 s_1 = s_2 s_1 s_2$):

$$\begin{aligned}
(1,j) &= s_1 s_2 s_3 s_4 \cdots s_{j-3} s_{j-2} s_{j-1} s_{j-2} s_{j-3} \cdots s_2 s_1 \\
&= s_1 s_2 s_3 s_4 \cdots s_{j-3} s_{j-1} s_{j-2} s_{j-1} s_{j-3} \cdots s_2 s_1 \\
&= s_{j-1} s_1 s_2 s_3 s_4 \cdots s_{j-3} s_{j-2} s_{j-1} s_{j-3} \cdots s_2 s_1 \\
&\vdots \\
&= s_{j-1} s_{j-2} s_{j-3} \cdots s_3 s_2 s_1 s_2 s_3 \cdots s_{j-3} s_{j-2} s_{j-1}
\end{aligned}$$

Localization

We used a technique called localization and reduced the proof to calculating this:

$$\sum_{U \text{ red. for } (1,j)} n_{W_{[m]}}(U)$$

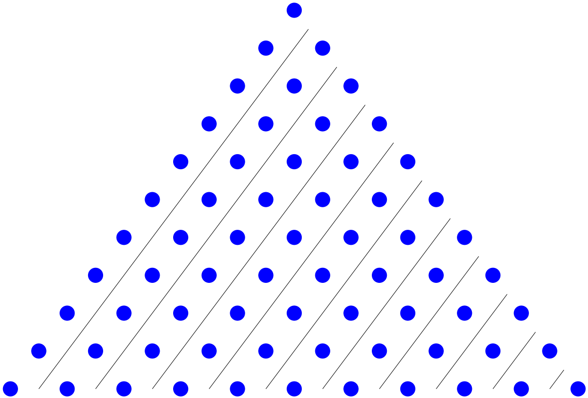
where

$$W_{[m]} = (s_1 s_2 \cdots s_{m-1})(s_1 s_2 \cdots s_{m-2}) \cdots (s_1 s_2) s_1,$$

we are summing over reduced words U for $(1, j)$, and $n_{W_{[m]}}(U)$ is the number of ways the reduced word U fits into $W_{[m]}$.

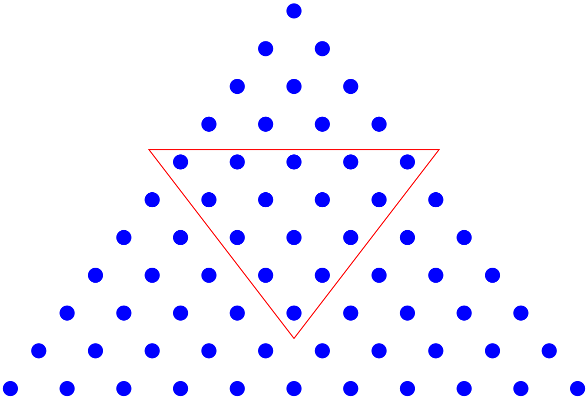
Triangle Representation of a Long Word

The following triangle represents w_{11}. A dot in row i represents simple reflection s_i. The j^{th} diagonal from the leftmost diagonal represents $s_1 s_2 \cdots s_{11-j}$.



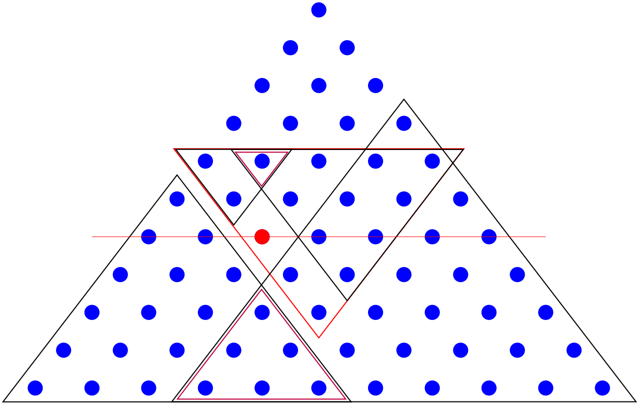
Triangle: Possible Braid Nodes

If we are restricting $(1, 8)$ to w_{11}, the nodes in the red triangle represent possible locations for the braid node. The possible braid indices are the integers between 3 and 7, inclusive.



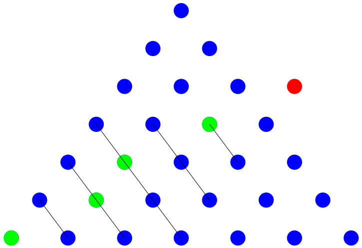
Triangle: Possible Ascending/Descending Subwords

Choosing one braid node, we get 4 triangles (black) that contain possible ascending and descending subwords. Simple reflections in subwords cannot correspond to dots in the purple intersections or in the tip of the triangle at and past the red line, which leaves a parallelogram.



Triangle: Choosing Diagonals

Given one parallelogram with dot height h and dot width w, we must place w dots corresponding to simple reflections in $h + w - 1$ diagonals; at most one corresponding dot can exist in one diagonal, and the location of the dot within a diagonal is implicit from the choice of diagonals.



Counting Elements of a Braid Index

By looking at all possible braid node choices on row $a \leq m$ on the triangle representation of w_m, we found that there are

$$Br(a,m) = \sum_{b=0}^{m-2j+a} \binom{j+b-1}{a-1} \binom{m+a-j-b-1}{a-1} \times \binom{j+b-a}{j-a} \binom{m-j-b}{j-a}$$

ways to write expressions of $(1, j+1)$ with braid index a as subwords of P_m. Denote the above expression by $Br(a,m)$. Then

$$\sum_U n_{w_m}(U) = \sum_{a=\max(2j-m,1)}^{j} Br(a,m).$$

Next Steps

- We want to simplify this expression.
- We want to be able to restrict transpositions to long words that don't necessarily start at 1.
- Afterwards, we'd want to be able to compute the pullback of these transpositions.