Persistent Homology

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Motivating Example

Lets consider the following data set.

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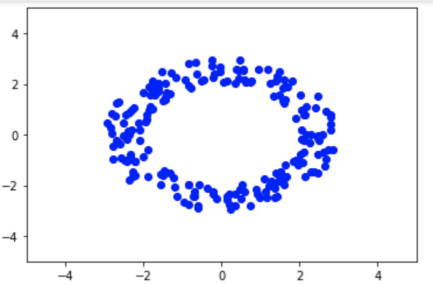

Figure: Plot of a noisy data set that represents a circle

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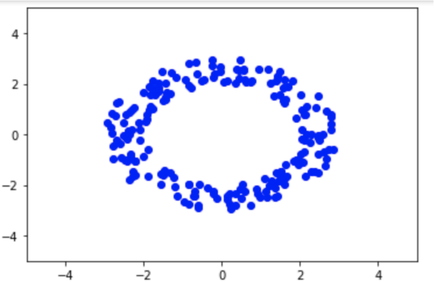


Figure: Plot of a noisy data set that represents a circle

Its clear that our data set has a specific qualitative feature. However, how can find this information computationally?

Simplical Complex

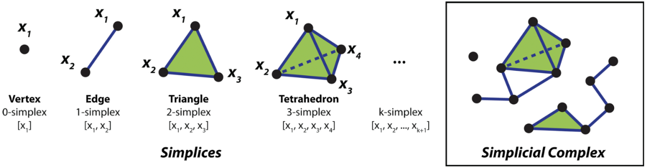
- Let $\{v_0, ..., v_k\}$ be a set of vectors in V. This set is said to be *convex independent* or *c-independent* if $\dim(\text{Span}\{v_0 - v_i, ..., v_k - v_i\}) = k$ for any $0 \leq i \leq k$.
- Let V be a vector space over $\mathbb{R}$. A convex set generated by c-independent vectors $\{v_0, v_1, \ldots, v_k\}$ is called a $k-$simplex
- We denote an open simplex as $(v_{i_1}, \ldots, v_{i_j})$ and a closed simplex by $[v_{i_1}, \ldots, v_{i_j}]$
- A simplical complex K (Euclidean) is a finite set of open simplices in some $\mathbb{R}^n$ such that

(1) if $(s) \in K$ then all open faces of $[s] \in K$;

(2) if $(s_1) \cap (s_2) \neq \emptyset$ then $(s_1) = (s_2)$ [1].

Simplical Complex (Continued)

Here is a visualization:

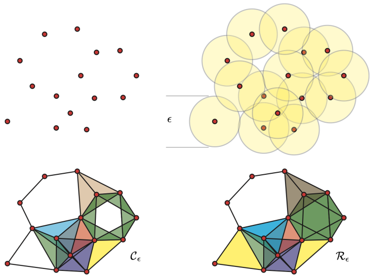


Notice that all the simplices here are closed. However, if we removed the vertices from the 1-simplex we have an open edge (x_1, x_2). Similarly, an open 2-simplex (x_1, x_2, x_3) by removing the vertices and edges. Etc.

(Image taken from [2])

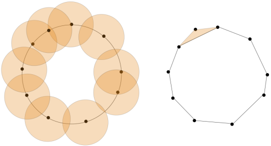
Computational Homology

- Cech Complex: Given a set of points $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^d$ and a real value $\epsilon > 0$,a k-simplex $\sigma = [p_{i0}, ..., p_{ik}]$ is in the Cech complex $\mathcal{C}_\epsilon(P)$ if and only if $\bigcap_{0 \le j \le k} \mathbb{B}(p_{i_j}, \epsilon) \neq \emptyset$ [3].
- Rips Complex: Given a set of points $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^d$ and a real value $\epsilon > 0$, a k-simplex $\sigma = [p_{i0}, ..., p_{i_k}]$ is in the Vietoris-Rips (Rips) complex $\mathcal{R}_\epsilon(P)$ if and only if $\mathbb{B}(p_{i_j}, \epsilon) \cap \mathbb{B}(p_{i_{j'}}, \epsilon) \neq \emptyset$ for any $j, j' \in [0, k]$.



(Image taken from [4])

Computational Homology (Continued)



- Consider the Cech complex, K_ϵ, above. It has a simplical chain complex

$$0 \xrightarrow{\partial_3} C_2(K_\epsilon, \mathbb{Z}) \xrightarrow{\partial_2} C_1(K_\epsilon, \mathbb{Z}) \xrightarrow{\partial_1} C_0(K_\epsilon, \mathbb{Z}) \xrightarrow{\partial_0} 0$$

where $C_i(K_\epsilon, \mathbb{Z})$ is an abelian group and ∂_i is a group homomorphism.

- The i-homology is defined to be $H_i(K_\epsilon, \mathbb{Z}) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1})$. In our case, $H_1(K_\epsilon, \mathbb{Z}) = \mathbb{Z} = H_0(K_\epsilon, \mathbb{Z})$.

- Futhermore, the i-betti number is given by the rank of the i-homology. These numbers are used to distinguish the difference between topological spaces.

Topological Simplifications

- Our algorithmic implementation this semester will primarily focusing on using the Rips Complex rather than the Cech Complex. This is because Cech Complex is computationally more expensive.
- In particular, the time complexity of the Cech Complex is $O(n^{k+1})$ where n is the number of points used and k is k-skeleton. Whereas the time complexity of the Rips Complex is $O(n^2)$.
- Another advantage for using the Rips Complex is that every k-simplex can be calculated solely by looking at the 1-Skeleton.

Topological Simplifications (Continued)

- Furthermore, since it is possible to calculate the homology of a simplicial complex over any free abelian group, we will be using $\mathbb{Z}_2$. This is because computing homology over $\mathbb{Z}$ with large data is incredibly inefficient.

- This allows the computation of the Betti numbers to be $\beta_i = \text{Rank}(H_i(K_\epsilon, \mathbb{Z}_2)) = \text{Rank}(\ker(\partial_i)) - \text{Rank}(\text{im}(\partial_{i+1}))$

- However, this does problematic when torsion is present. In the absence of torsion, the Betti numbers under $\mathbb{Z}_2$ are the same as those under $\mathbb{Z}$, according to the Universal Coefficient Theorem [5]. We may still use $\mathbb{Z}_2$ coefficients in the presence of torsion, but these answers may differ from those computed using $\mathbb{Z}$ coefficients.

Algorithmic Implementation

As stated before, in order to create the Rips Complex for a given set of data it is sufficient to compute the 1-Skeleton. This is done by the following algorithm

Algorithm 1 Skeleton of the Rips Complex

Ensure: $\epsilon > 0$ and $P_n \neq \emptyset$

```
edges = []
for i in P_n do
    for j in P_n \ i do
        if d(i, j) > 2ε then
            continue
        else
            edges.append(⟨i, j⟩)
        end if
    end for
end for
```

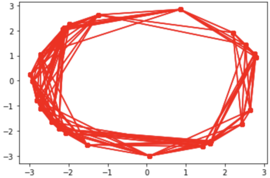
Algorithmic Implementation (Continued)

Now that we have the 1-Skeleton, the information pertaining to the entire complex is implicitly encoded. In order to uncover the higher simplicies we will need to recursively build on top of the 1-Skeleton.

This can be done by building two functions: a simplicial builder and simplicial decomposition function.

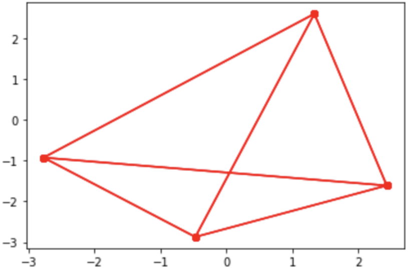
Example

```
In [8]: ### Dataset ###

        Class_1 = circle(2.9,3,0,0,25)


        ### The Skeleton ###

        x = rc_one_simplex(Class_1, 2)

        ### For smaller complex set size < len(x). ###
        ### Else, for the entire simplical complex set times = len(x). ###

        RC = Rips_Complex(len(x),x)
```



Here is the 1-Skeleton of a similar dataset from the beginning of the example.

Computational Example




```
[[[0, 1], [0, 2], [0, 3], [1, 2], [1, 3], [2, 3]], [[0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3]], [[0, 1, 2, 3]]]
```

Now that we have the entire simplicial complex we can create the boundary maps and find the betti numbers.

Future Work

- The next goal would be to finish the algorithm for computing the boundary matrix and find the dimension of the image and kernel.
- Currently we have the ability to create the correct boundary maps. However, we cannot use classic row reduction methods since our matrices are over $_2$.
- We would also like to create persistent diagrams of the betti numbers as a function of time.

Citations

[1] Singer, Isadore Manuel, and John A. Thorpe. Lecture notes on elementary topology and geometry. Springer, 2015.

[2] Zhang, Mengsen Kalies, William Kelso, Scott Tognoli, Emmanuelle. (2020). Topological portraits of multiscale coordination dynamics. Journal of Neuroscience Methods. 339. 108672. 10.1016/j.jneumeth.2020.108672.

[3] Otter, Nina, et al. "A roadmap for the computation of persistent homology." EPJ Data Science 6 (2017): 1-38.

[4] Robert Ghrist. Barcodes: the persistent topology of data. Bull. Amer. Math. Soc. (N.S.), 45(1):61–75, 2008. ISSN 0273-0979.

[5] Edelsbrunner, Herbert, David Letscher, and Afra Zomorodian. "Topological persistence and simplification." Proceedings 41st annual symposium on foundations of computer science. IEEE, 2000.