Cores and Hulls of Ideals of Commutative Rings

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Rings

Definition

A **ring** is a set R equipped with two operations "addition" $+$ and "multiplication" $\cdot$ where:

- R is closed under addition
- R has an additive identity 0_R
- R contains additive inverses for all $r \in R$
- $+$ is associative and commutative
- R is closed under multiplication
- $\cdot$ is associative (and has identity 1)
- $\cdot$ distributes over $+$

If multiplication is commutative, R is a commutative ring.

Examples of Commutative Rings

- Number Rings: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$
- Polynomial Rings: $\mathbb{Z}_2[x]$, $\mathbb{Q}[x, y]$

Ideals of Commutative Rings

Definition

A subset I of a commutative ring R is an **ideal** of R if:

- $0_R \in I$
- I is closed under same addition as R
- Every element in I has an additive inverse contained in I
- $r \cdot i \in I \ \forall \ i \in I$ and $r \in R$

The set $(f_1, \dots, f_s) = \{a_1 f_1 + \cdots + a_s f_s \mid a_i \in R\}$ is the **ideal generated by** $f_1, \dots, f_s$. Furthermore, if I is an ideal of a ring R, then R/I is a ring where the elements of I act as 0_R.

Examples

- In the quotient ring $k[x_1, x_2, x_3]/(x_1 x_3)$ we set $x_1 x_3 = 0$. As a consequence, $x_1 \in (x_3 + 1)$ since $x_1 = x_1(x_3 + 1)$ and $x_3 + 1 \in (x_3 + 1)$
- For the above ring, $(x_1 + x_3)^n = x_1^n + x_3^n$

Definition

A prime ideal $P \subseteq R$ is an ideal such that $ab \in P$ implies $a \in P$ or $b \in P$. It is minimal if P contains no other prime ideals.

Example

For every prime number $p \in \mathbb{Z}$, (p) is a prime ideal of $\mathbb{Z}$ since $p|ab$ implies $p|a$ or $p|b$

Special Ideals In Commutative Rings

Definition

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Definition

The Annihilator of an ideal is the ideal

$$Ann_R(I) = \{r \in R \mid r \cdot i = 0_R \text{ for all } i \in I\}$$

Example in $S = k[x_1, x_2, x_3]/(x_1x_3)$

The minimal prime ideals of S are (x_1) and (x_3). Since $x_1x_3 = 0$, $Ann_R(x_1) = (x_3)$ and $Ann_R(x_3) = (x_1)$.

Simplicial Complexes In Commutative Algebra

Definition

An **(abstract) simplicial complex** is a collection Δ of subsets of $\{x_1, \ldots, x_n\}$ (called faces) such that

- If F is a face of Δ, and S is any nonempty subset of F, then S is a face of Δ.
- For any two faces F_1, F_2 of Δ, $F_1 \cap F_2$ is also a face of Δ.

For our purposes we will take note of non-faces of Δ.

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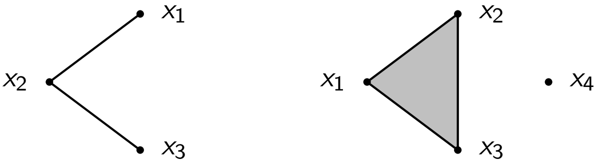

Figure: An illustration of two different simplicial complexes. We note that for the one on the left, $\{x_1, x_3\}$ and $\{x_1, x_2, x_3\}$ are non-faces

Stanley-Reisner Rings

Definition

- **Stanley-Reisner rings** are a class of quotient rings found in combinatorial commutative algebra. If k is a prime-characteristic field, and I is a square-free monomial ideal (generated by products of variables of degree 1) in $k[x_1, \ldots, x_n]$, then $S = k[x_1, \ldots, x_n]/I$ is a Stanley-Reisner Ring. In this ring, the polynomials of I behave as zero ring elements.
- For any simplicial complex, we have an associated Stanley-Reisner ring generated by the non-faces of our complex.

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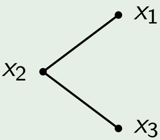
Example



$$\longrightarrow \qquad S = \frac{k[x_1, x_2, x_3]}{(x_1 x_3)}$$

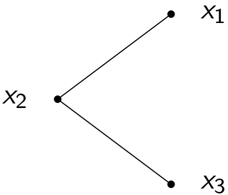


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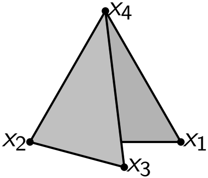
Figure: Our 'base' simplicial complex for computing examples



$\longrightarrow \quad \frac{k[x_1, \ldots, x_4]}{(x_1 x_3)}$

Figure: A simplicial complex with the same relations on its Stanley-Reisner ring, but has an additional vertex.

Interior Operations

Definition

Let I and J be ideals of a ring R. An operation
int : Ideals of R $\rightarrow$ Ideals of R is called an **interior operation** if:

- $I_{int} \subseteq I$
- $(I_{int})_{int} = I_{int}$
- For $I \subseteq J$, $I_{int} \subseteq J_{int}$

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Example

The mapping that takes any ideal in R to the zero ideal is an interior operation.

$$I_{int} = (0)$$

The interior operation we are studying is called the tight interior

Tight Interiors and $* - hull$

Theorem (Vassilev 2021)

Let $P_1, \ldots, P_m$ be the minimal prime ideals of a Stanley-Reisner ring S. Then the tight interior of an ideal $I \subseteq S$ is

$$I_* = \sum_{i=1}^{m} Ann_R(P_i) \cap I$$

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Definition

Let I be an ideal of a ring R. Then the $* - hull$ of I in R is the ideal

$$* - hull(I) = \sum_{I \subseteq J, I_* = J*} J$$

Where the $J's$ are called $*$-expansions. Notice we need only sum over the maximal $*$-expansions (i.e. expansions not contained in any other) to find the $* - hull$.

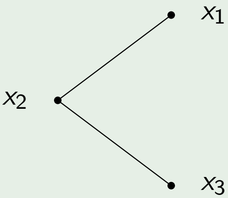
Results

When is $* - hull(I) = S$?

- The minimal prime ideals of $S = k[x_1, x_2, x_3]/(x_1x_3)$ are (x_1) and (x_3)

$$I_* = (x_1) \cap I + (x_3) \cap I,$$

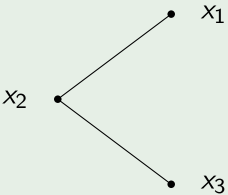
  for any ideal.
- Therefore, $S_* = (x_1) + (x_3) = (x_1, x_3)$ and any ideal with the same interior will have S as its hull.
- Therefore, if $x_1, x_3 \in I$, then $* - hull(I) = S$.



Results

Results in $k[x_1, x_2, x_3]/(x_1x_3)$

1. $I_* = (x_1) \cap I + (x_3) \cap I$.
2. If $x_1, x_3 \in I$, then $I_* = (x_1, x_3)$ and $* - hull(I) = S$.
3. If $p \in (x_1)$ or $p \in (x_3)$, then $(p)_* = (p)$.
4. $(x_2, x_3)_* = (x_1x_2, x_3)$ where (x_2, x_3) is a maximal $*$-expansion for (x_1x_2, x_3).
5. Many specific examples: $(x_3 + 1)_* = (x_1, x_3^2 + x_3)$, $(x_2)_* = (x_1x_2, x_2x_3)$, etc.



Theorem (General Gluing Rule)

Let $\Delta = \Delta_1 \cup_{p_1,\dots,p_d} \Delta_2$ be the resulting simplicial complex from gluing the simplicial complexes Δ_1 and Δ_2 along the common non-isolated (in both Δ_1 and Δ_2) points $p_1, \dots, p_d$. If the vertices of Δ_1 are $x_1, \dots, x_m, p_1, \dots, p_d$ and the vertices of Δ_2 are $y_1, \dots, y_n, p_1, \dots, p_d$, then we have

$$I_*^{(S)} = I_*^{(S_1)} \cap (x_1, \dots, x_m) + I_*^{(S_2)} \cap (y_1, \dots, y_n). \tag{1}$$

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Theorem (Tight Interiors of Trees Classification)

Let Δ be a tree that is not the line segment with endpoints $p_1, \ldots, p_m$. Then

$$I_* = I \cap (p_1, \ldots, p_m).$$

Example Tree

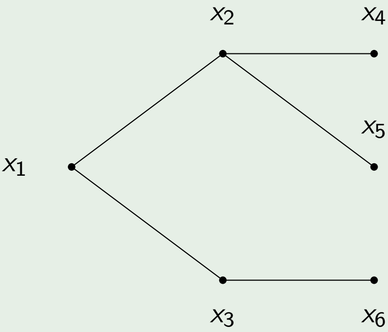

Figure: Our tree example

$$S = k[x_1, \ldots, x_6]/(x_1x_4, x_1x_5, x_1x_6, x_2x_3, x_2x_6, x_3x_4, x_3x_5, x_4x_5, x_4x_6, x_5x_6).$$

We have $I_* = I \cap (x_4, x_5, x_6)$ by the tight interiors of trees classification.

Future Explorations

- Compute more results for simple example and extend them to rings with more variables.
- Use the results from this semester to classify interiors for more complicated Stanley-Reisner rings.
- Relate 1st homology of a complex to interiors and hulls.

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