Cores and Hulls of Ideals of Commutative Rings

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Stanley-Reisner Rings

Stanley-Reisner rings are a class of quotient rings found in combinatorial commutative algebra. If I is a square-free monomial ideal (generated by product of variables with degree 1) in $k[x_1, \dots, x_n]$, then $S = k[x_1, \dots, x_n]/I$ is a Stanley-Reisner Ring. In a Stanley-Reisner ring, every polynomial previously in I now behaves as the zero ring element.

Crucial in understanding Stanley-Reisner rings are the simplicial complexes associated to each Stanley-Reisner ring. We used this association to compute examples of tight interiors for ideals of S and sought to generalize these results by studying the effect adjusting the complex had on interiors and hulls.

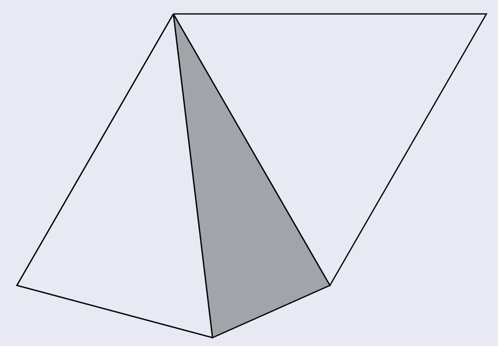
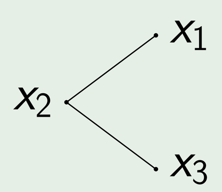


Figure: Simplicial complexes can be constructed from n-dimensional tetrahedra.

Example Ring

By considering the minimal non-faces of Δ given below, we can make the following association:



$$\longrightarrow \qquad k[\Delta] = \frac{k[x_1, x_2, x_3]}{(x_1 x_3)}$$

In the ordinary polynomial ring, there cannot exist $g \in k[x_1, x_2, x_3]$ such that $g \cdot (x_3 + 1) = x_1$. This does not hold in $k[\Delta]$ since $x_1(x_3 + 1) = x_1 x_3 + x_1 = 0 + x_1 = x_1$.

Ideals of Commutative Rings

A subset I of a commutative ring R is an **ideal** of R if:

- $0_R \in I$
- I is closed under same addition as R
- Every element in I has an additive inverse contained in I
- $r \cdot i \in I \ \forall \ i \in I$ and $r \in R$

Finitely Generated Ideals

The set of all elements in $r \in R$ such that $r = a \cdot b$ for some $b \in R$ is called the ideal generated by a and is written (a). Given n ideals $(a_1), (a_2), \dots, (a_n)$, $(a_1, a_2, \dots, a_n)$ is the smallest ideal containing all (a_i).

Interior Operations

Let I and J be ideals of a ring R. An operation *int* : Ideals of R $\rightarrow$ Ideals of R is called an **interior operation** if:

- $I_{int} \subseteq I$
- $(I_{int})_{int} = I_{int}$
- For $I \subseteq J$, $I_{int} \subseteq J_{int}$

Our research focuses on the tight interior operation, or I_*.

*-hulls

Let I_S be the set of ideals J with $I \subseteq J$, $J_* = I_*$. Then,

$$* - hull(I) = \sum_{J_i \in I_S} J_i$$

We say that all J_i are ***-expansions** of I. We can define *int*-hulls for other interior operations, but we only focused on the *-hull this semester.

Abstract Simplicial Complexes

An **abstract simplicial complex** is a collection Δ of subsets of $\{x_1, \dots, x_n\}$ (called faces) such that

- If F is a face of Δ, and F' is any nonempty subset of F, then F' is also a face of Δ
- For any two faces F_1, F_2 of Δ, $F_1 \cap F_2$ is also a face of Δ

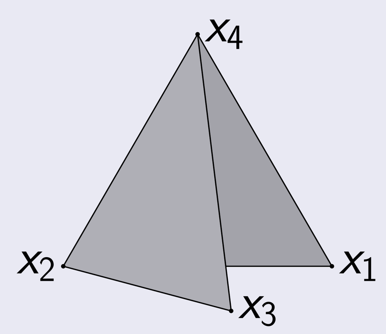


Figure: Example of a simplicial complex. An edge or filled in face represents the connected vertices being in the same face of Δ.

Theorem (Vassilev 2021) - Tight interior

Let $P_1, \dots, P_m$ be the minimal prime ideals of a Stanley-Reisner ring S. Then the tight interior of an ideal $I \subseteq S$ is

$$I_* = \sum_{i=1}^{m} Ann_R(P_i) \cap I$$

Theorem 1 (Results in $k[x_1, x_2, x_3]/(x_1 x_3)$)

1. $I_* = (x_1) \cap I + (x_3) \cap I$.
2. If $x_1, x_3 \in I$, then $I_* = (x_1, x_3)$ and $* - hull(I) = S$.
3. If $p \in (x_1)$ or $p \in (x_3)$, then $(p)_* = (p)$.
4. $(x_2, x_3)_* = (x_1 x_2, x_3)$ where (x_2, x_3) is a maximal $*$-expansion for $(x_1 x_2, x_3)$.
5. Many specific examples: $(x_3 + 1)_* = (x_1, x_3^2 + x_3)$, $(x_2)_* = (x_1 x_2, x_2 x_3)$, etc.

Theorem 2

Let $\Delta = \Delta_1 \sqcup \Delta_2$ be a disjoint union of two simplicial complexes and $S_i = k[\Delta_i]$. Let $I_*^{(S_i)}$ denote the computations of tight interior for $I \subset S$ as if it were an ideal in a Stanley-Reisner ring $k[\Delta_i]$. That is, the expression $I_*^{(S_1)}$ is the extension of the tight interior formula for S_1 extended to ideals in $S = k[\Delta]$. Then we have $I_*^{(S)} = I_*^{(S_1)} + I_*^{(S_2)}$.

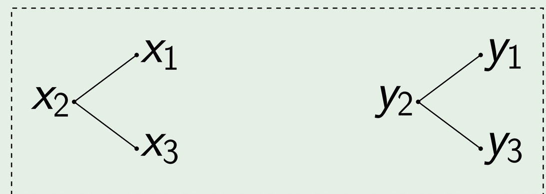


Figure: Disjoint union of identical simplical complexes.

Ex: Let $I = (x_2, x_3)$. Then applying Theorem 1, we obtain

$$\begin{aligned} I_*^{(S)} = I_*^{(S_1)} + I_*^{(S_2)} &= (x_1 x_2, x_2 x_3) + (y_1 x_2, y_2 x_3) \\ &= (x_1 x_2, x_2 x_3, y_1 x_2, y_2 x_3) \end{aligned}$$

Theorem 3

Let $\Delta = \Delta_1 \cup_p \Delta_2$ be the resulting simplicial complex from gluing the simplicial complexes Δ_1 and Δ_2 at a common non-isolated (in both Δ_1 and Δ_2) point p, i.e. taking the wedge product. Using the notation from before, if the vertices of Δ_1 are $x_1, \dots, x_m, p$ and the vertices of Δ_2 are $y_1, \dots, y_n, p$, then we have

$$I_*^{(S)} = I_*^{(S_1)} \cap (x_1, \dots, x_m) + I_*^{(S_2)} \cap (y_1, \dots, y_n).$$

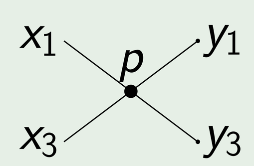


Figure: Simplicial complex obtained by letting $x_2 = y_2 = p$ in the previous example.

Example Interior

Consider the ideal (p) in Δ from Theorem 3. The two pieces of Δ are identical to the complex used in Theorem 1. Therefore, $I_*^{(S_1)} = (x_1) \cap I + (x_3) \cap I$ and $I_*^{(S_2)} = (y_1) \cap I + (y_2) \cap I$. Thus, $(p)_*^{(S_1)} = (px_1, px_3)$ and $(p)_*^{(S_2)} = (py_1, py_3)$. Using Theorem 3 we obtain

$$(p)_*^{(S)} = (p)_*^{(S_1)} \cap (x_1, x_3) + (p)_*^{(S_2)} \cap (y_1, y_3) = (px_1, px_2, py_1, py_2).$$

Theorem 4

Let Δ be a tree that is not the line segment with endpoints $p_1, \dots, p_m$. Then

$$I_* = I \cap (p_1, \dots, p_m).$$

Theorem 5

Let $\Delta = \Delta_1 \cup_{p_1, \dots, p_d} \Delta_2$ be the resulting simplicial complex from gluing the simplicial complexes Δ_1 and Δ_2 along the common non-isolated (in both Δ_1 and Δ_2) points $p_1, \dots, p_d$. If the vertices of Δ_1 are $x_1, \dots, x_m, p_1, \dots, p_d$ and the vertices of Δ_2 are $y_1, \dots, y_n, p_1, \dots, p_d$, then we have

$$I_*^{(S)} = I_*^{(S_1)} \cap (x_1, \dots, x_m) + I_*^{(S_2)} \cap (y_1, \dots, y_n).$$

Future Explorations

- Calculate more examples of hulls for ideals in Stanley-Reisner rings.
- Use the results from this semester to classify interiors for more complicated Stanley-Reisner rings.
- Relate the homology of a complex to interiors and hulls.

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