Combinatorial Formulas for the Equivariant Cohomology of Peterson Varieties

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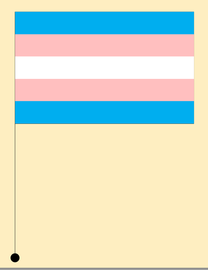
Introduction

- We look at a variety X and a subvariety Y and consider a circle S acting on X, under which Y is invariant.
- We associate to each space a graded ring.
- Given $Y \overset{\iota}{\hookrightarrow} X$, there is a natural induced surjective map $\iota^* : H_S^*(X) \to H_S^*(Y)$ which we want to describe.
- $H_S^*(X)$ and $H_S^*(Y)$ each have a module basis we want to explore.

Complete Flag Variety

$$X = Fl(\mathbb{C}^n) = \{0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n | \dim(V_i) = i\}$$

Each point in X is a chain of vector spaces.



$$\mathbb{R}^0 \subseteq \mathbb{R}^1 \subseteq \mathbb{R}^2$$

Peterson Variety

The Peterson variety Y is the collection of complete flags satisfying the condition $MV_i \subset V_{i+1}$ for $1 \leq i \leq n-1$ where M is a principal nilpotent operator.

Equivariant Cohomology

S acts on Y with isolated fixed points, indexed by subsets $A \subseteq [n-1]$.

Equivariant cohomology $H_S^*(Y)$ can be regarded as a subring of

$$\bigoplus_{A \subseteq [n-1]} \mathbb{Q}[t].$$

Every equivariant cohomology class is then represented by an 2^{n-1}-tuple of polynomials.

Schubert classes on X and on Y

Basis for $H_S^*(X)$: Schubert classes σ_v, indexed by elements of S_n.

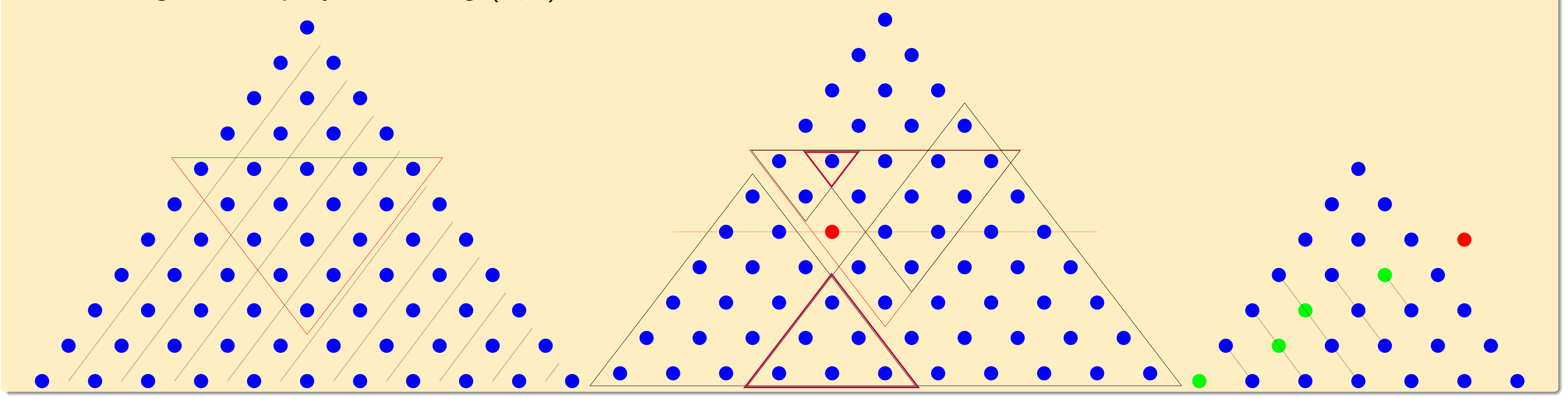
Basis for $H_S^*(Y)$: Peterson classes, p_I each indexed by $I \subseteq [n] = \{1, 2, \cdots n\}$. Peterson classes are all images of specific Schubert classes under ι^*.

Goal

The goal is to express the restriction of transposition Schubert classes to the Peterson variety as a linear combination of Peterson classes.

Combinatorial Triangles

These triangles exemplify restricting $(1, 8)$ to w_{11}.



Main Tool: Localization

Useful Lemma: Let $\sigma_u \in H_T^*(X)$ be a Schubert class and let w_A be the S-fixed point of the Peterson variety Y associated to $A \subseteq \{1, ..., n-1\}$. Let W_A be reduced-word representation for w_A of the following form: For each continuous subset of A, without loss of generality $\{a, a+1, \cdots b\}$, we multiply $(s_a s_{a+1} \cdots s_{b-1})(s_a \cdots s_{b-2}) \cdots (s_a s_{a+1}) s_a$. Then

$$\iota^*(\sigma_u)|_{w_A} = \sum_{U \in \rho(u)} n_{W_A}(U) \left(\prod_{j \in U} (j - \mathcal{T}_A(j) + 1) \right)$$

where $\rho(u)$ is the set of reduced words of u, $n_{W_A}(U)$ is the number times the word U occurs as a subword of W_A, and $\mathcal{T}_A(j)$ is the smallest integer in the maximal consecutive subset of A containing j.

Pullback Linear Combinations and the Peterson Class Basis

We are interested in restricting elements of X to Y to obtain the Schubert classes corresponding to each element. We then can find the pullback $\iota^*(\sigma_v)$ for each element $v \in X$ and write the result as a linear combination of Peterson classes, $\iota^*(\sigma_c)$ for each $c \in C$.

Equivariant Cohomology

$$\begin{tikzcd}
H_S^*(X) \arrow[r, "\iota^*"] \arrow[d] & H_S^*(Y) \arrow[d] \\
H_S^*((X)^S) \arrow[r, "\iota^*_{fps}"] \arrow[d, Rightarrow] & H_S^*(Y^S) \arrow[d, Rightarrow] \\
\bigoplus_{w \in W} H_S^* \arrow[r] & \bigoplus_{w_A \in S_n} H_S^*
\end{tikzcd}$$

Subword Counts:

$c_j = s_1 s_2 \cdots s_j$ is a subword of w_m where $w_m = (s_1 s_2 \cdots s_m)(s_1 \cdots s_{m-1}) \cdots (s_1 s_2) s_1$ in $\binom{m}{j}$ different ways.

Let

$$Br(a, m) = \sum_{b=0}^{m-2j+a} \binom{j+b-1}{a-1} \binom{m+a-j-b-1}{a-1} \binom{j+b-a}{j-a} \binom{m-j-b}{j-a}.$$

Then, we have that

$$\sum_U n_{W_{[m]}}(U) = \sum_{a=\max(2j-m,1)}^{j} Br(a, m).$$

Conjecture

Let (i, j) be the transposition of i and j, where $i < j$, and call $m \equiv j - i$ the magnitude of the transposition. We have that

$$\iota^*(\sigma_{(i,j)}) = \sum_{k=0}^{m-1} \sum_{h=0}^{k} h! \binom{k}{h}^2 \binom{m-1}{k}^2 t^h p_{\{1+i+k-m, \cdots, j+k-h-1\}}$$

excluding terms where $1 + i - k - m < 1$ or $j + k - h \geq n$. For $(1, j)$, this is equivalent to stating that for all m

$$\binom{m}{j} \binom{m-1}{j-1} = \sum_{U \in \rho((i,j))} n_{W_{[m]}}(U).$$

Combinatorial Insight

We represent $W_{[m]}$ as a triangle of dots, each row representing a specific transposition, the bottom s_1, the next s_2, etc., and each string $s_1 s_2 \cdots s_{m-b}$ corresponding to the b^{th} up and right diagonal from the left. Any reduced word for $(1, j)$ has a *braid index*, the index of the middle transposition, which is also the only transposition present only once in the reduced word. This middle transposition can occupy only certain spaces in $W_{[m]}$, represented by the large inverted red triangle. The other transpositions in a reduced word break into four independent groups, those in the large upright and inverted black triangles. Each individual group is either ascending or descending, and to count how many ways to fit reduced words with a specific middle transposition into $W_{[m]}$, the number for each of the four groups give binomial coefficients. Summing over all the middle transpositions with the braid index a, we get $Br(a, m)$.

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