Stability of Traveling Waves in Nonlocal Diffusion Models

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Outline

- Equilibria and stability
- Linear operators and spectrum
- Semigroup operator and bounds
- Our project
- Next semester

General Differential Equations

Consider an equation

$$u_t = \mathcal{F}(u),$$

where

- $u = u(x,t) \in X$, for some function space X,
- $\mathcal{F}(u)$ contains no time derivatives.

An **equilibrium** is a solution u^* such that $u_t^* = \mathcal{F}(u^*) = 0$.

Stability

We wish to determine the **stability** of u^*.

We do this by adding a small **perturbation** p and analyzing the solution

$$u(x,t) := u^*(x) + p(x,t).$$

If $u \to u^*$ (i.e. $p \to 0$) as $t \to \infty$, then u^* is **stable**.

We do a Taylor series expansion about u^*:

$$(u^* + p)_t = \mathcal{F}(u^* + p),$$

$$u^*_t + p_t = \mathcal{F}(u^*) + \mathcal{F}'(u^*)p + \frac{1}{2}\mathcal{F}''(u^*)p^2 + \dots,$$

$$p_t = \mathcal{L}p + \mathcal{N}(p),$$

where $\mathcal{L} := \mathcal{F}'(u^*)$ and $\mathcal{N}(p)$ is the higher order terms.

Linear Stability

$$p_t = \mathcal{L}p + \mathcal{N}(p),$$

When p is really small, we expect $\mathcal{L}p$ to dominate.

Thus, the solution to $p_t = \mathcal{L}p$ should tell us what happens to p.

In MATH 214, we deal with the case where $\mathcal{L}$ is a **matrix**.

In this case, every solution can be constructed from **eigenvectors**, special (nonzero) solutions such that

$$\mathcal{L}p = \lambda p,$$

for some $\lambda \in \mathbb{C}$, called an **eigenvalue**.

Spectrum of a Matrix

Notice

$$\mathcal{L}p = \lambda p,$$

$$\mathcal{L}p - \lambda p = 0,$$

$$(\mathcal{L} - \lambda I)p = 0,$$

and hence $(\mathcal{L} - \lambda I)$ is not invertible.

We define the **spectrum** of $\mathcal{L}$ to be

$$\sigma(\mathcal{L}) := \{\lambda \in \mathbb{C} \mid (\mathcal{L} - \lambda I)^{-1} \text{ is not defined}\}.$$

We call $(\mathcal{L} - \lambda I)^{-1}$ the **resolvent** operator.

However, these concepts don't just apply to matrices.

Spectrum of an Operator

For example, consider the linear operator $\mathcal{L} : L^\infty \to L^\infty$ defined by

$$\mathcal{L}f := \partial_x^2 f = f_{xx}.$$

An **eigenfunction** of $\mathcal{L}$ is a function $f \in L^\infty$ such that

$$\mathcal{L}f = \lambda f,$$

for some eigenvalue $\lambda \in \mathbb{C}$.

Similar to before, the **spectrum** of $\mathcal{L}$ is

$$\sigma(\mathcal{L}) := \{\lambda \in \mathbb{C} \mid (\mathcal{L} - \lambda)^{-1} \text{ is not defined or is unbounded}\}.$$

Although, not everything in $\sigma(\mathcal{L})$ is an eigenvalue.

For $\lambda = -1$ and $g(x) = \cos x$, we have

$$(\mathcal{L} - \lambda)^{-1} g = f \quad \Longrightarrow \quad f_{xx} + f = \cos x \quad \Longrightarrow \quad f(x) = f_h(x) + \frac{x}{2}\sin x$$

Laplace Transform

Recall the **Laplace Transform** from MATH 214:

$$\tilde{p}(x,\lambda) := \int_0^\infty e^{\lambda t} p(x,t)dt, \qquad \lambda \in \mathbb{C}$$

and the fact $\tilde{(p_t)} = \lambda\tilde{p} - p_0$, where $p_0(x) = p(x,0)$.

The **Inverse Laplace Transform** is given by

$$p(x,t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\tilde{p}(x,\lambda)d\lambda,$$

where Γ is a contour in $\mathbb{C}$ that extends to infinity.

Pertubation Solution

Hence

$$p_t = \mathcal{L}p,$$

$$\lambda\tilde{p} - p_0 = \mathcal{L}\tilde{p},$$

$$(\mathcal{L} - \lambda)\tilde{p} = p_0,$$

$$\tilde{p} = (\mathcal{L} - \lambda)^{-1} p_0,$$

$$p(x,t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\mathcal{L} - \lambda)^{-1} p_0(x) d\lambda.$$

Note that Γ must avoid $\sigma(\mathcal{L})$ so that the integrand is defined.

We define the **semigroup operator**

$$e^{t\mathcal{L}} := \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\mathcal{L} - \lambda)^{-1} d\lambda$$

so that $p = e^{t\mathcal{L}} p_0$.

Norm of an Operator

If $\mathcal{L}$ is a linear operator, the **operator norm** of $\mathcal{L}$ is

$$\|\mathcal{L}\| := \inf\{k > 0 : |\mathcal{L}u| \leq k|u| \text{ for all } u \in X\},$$

where $|\cdot|$ is the norm on X.

$\|\mathcal{L}\|$ is maximum "stretching factor" of $u \in X$,

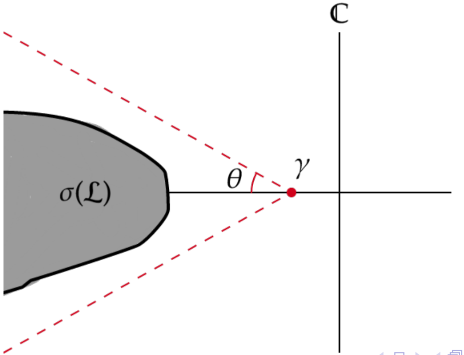
i.e. $|\mathcal{L}u| \leq \|\mathcal{L}\||u|$ for any $u \in X$.

Sectorial Operator

A linear operator $\mathcal{L}$ is **sectorial** if $\sigma(\mathcal{L})$ lies in some sector of $\mathbb{C}$ and

$$\|(\mathcal{L}-\lambda)^{-1}\| \leq \frac{M}{|\lambda-\gamma|},$$

for some $M > 0$, where $\gamma \in \mathbb{R}$ is the vertex of the sector.



Semigroup Bounds

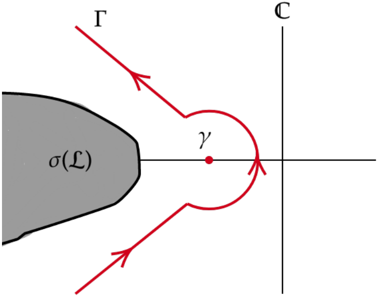
Theorem

If $\mathcal{L}$ is sectorial operator with vertex γ, then

$$\|e^{t\mathcal{L}}\| \leq Me^{\gamma t}$$

for some $M > 0$.

$$e^{t\mathcal{L}} := \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\mathcal{L}-\lambda)^{-1}d\lambda$$



Perturbation Bounds

Recall

$$p = e^{t\mathcal{L}} p_0.$$

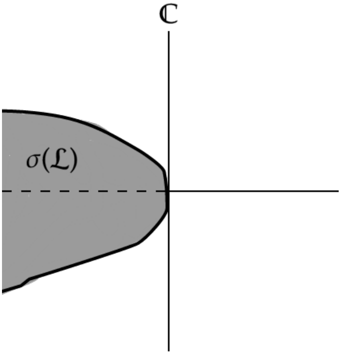
Hence

$$|p| \leq \|e^{t\mathcal{L}}\| |p_0| \leq Me^{\gamma t} |p_0|.$$

If we can choose $\gamma < 0$, then $p \to 0$ as $t \to \infty$, just as we wanted!

Otherwise, there is no **spectral gap**, and finding a bound for $|p|$ is trickier.

We need to use **pointwise estimates** to get more specific bounds on $\|e^{\mathcal{L}t}\|$.



Our Equation

The Fisher-KPP equation models population growth and spread:

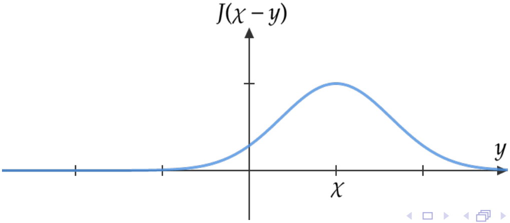
$$u_t = u_{\chi\chi} + f(u), \quad f(u) = \beta u(1-u)$$

We add a **nonlocal** diffusion term:

$$u_t = u_{\chi\chi} + (J * u - u) + f(u),$$

where

$$J * u(\chi) := \int_{-\infty}^{\infty} J(\chi - y)u(y)dy, \quad J(\chi) = \frac{1}{2\alpha} e^{-\alpha|\chi|} \text{ for some } \alpha > 0.$$

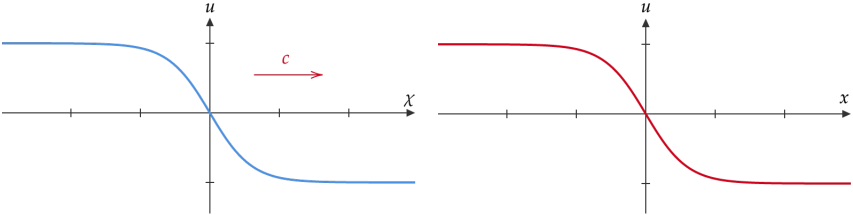


Wave Solutions

Consider a coordinate frame moving right with speed c:

$$x = \chi - ct$$

An equilibrium solution $u^*(x)$ in the (x, t) frame is a **traveling wave** in the (χ, t) frame with speed c.



Change of Coordinates

Using the chain rule, we can rewrite in terms of x:

$$u_t = u_{\chi\chi} + (J * u - u) + f(u)$$

$$u_t - cu_x = u_{xx} + (J * u - u) + f(u)$$

$$u_t = u_{xx} + cu_x + J * u - u + f(u)$$

Equilibria to this equation are well studied.

We wish to study their stability.

Perturbing the Equilibrium

Let $u(x,t) := u^*(x) + p(x,t)$. Then

$$(u^* + p)_t = (u^* + p)_{xx} + c(u^* + p)_x + J * (u^* + p) - (u^* + p) + f(u^* + p)$$

$$\begin{array}{ccl} u^*_t & = & u^*_{xx} + cu^*_x + J * u^* - u^* + f(u^*) \\ +p_t & & +p_{xx} + cp_x + J * p - p + f'(u^*)p + \frac{1}{2} f''(u^*)p^2 \end{array}$$

$$p_t = p_{xx} + cp_x + J * p - p + f'(u^*)p + \frac{1}{2} f''(u^*)p^2$$

$$p_t = \mathcal{L}p + \mathcal{N}(p)$$

where

$$\mathcal{L} = \partial_x^2 + c\partial_x + J * -1 + f'(u^*)$$

and $\mathcal{N}(p) = \frac{1}{2} f''(u^*)p^2$.

Localizing the Kernel

$$\mathcal{L} = \partial_x^2 + c\partial_x + J * -1 + f'(u^*)$$

Recall $J(\chi) = \dfrac{1}{2\alpha} e^{-\alpha|\chi|}$.

Let

$$\omega := J * p = \int_{-\infty}^{\infty} J(\chi - y)p(y)dy.$$

We can show

$$\omega_{xx} = \alpha^2\omega - p.$$

$$\begin{cases} p_t = p_{xx} + cp_z + \omega - p + f'(u^*)p \\ 0 = \omega_{xx} - \alpha^2\omega + p \end{cases}$$

Next Semester

1. Compute $\sigma(\mathcal{L})$ using the **Fourier Transform**:

$$\hat{p}(k,t) := \int_{-\infty}^{\infty} e^{-ikt} p(x,t)dx, \qquad k \in \mathbb{R}$$

2. Since there is no spectral gap, we will obtain **pointwise estimates** of $e^{\mathcal{L}t}$ using a **Green's function**:

$$p(x,t) = \int_{\infty}^{\infty} \mathcal{G}(x,y,t)p(y,0)dy$$

$$\mathcal{G}(x,y,t) = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t} G(x,y,\lambda)d\lambda$$

3. Determine linear stability from estimates on $e^{\mathcal{L}t}$.

4. Analyze **nonlocal stability**.

References

[1] M. Beck, "Stability of nonlinear waves: pointwise estimates," *Discrete & Continuous Dynamical Systems*, vol. 10, no. 2, pp. 191-211, 2017

[2] M. Beck, "Topics in Stability Theory for Partial Differential Equations," *Boston College*, 2006

[3] J. Carr and A. Chmaj, "Uniqueness of Travelling Waves for Nonlocal Monostable Equations," *Proceedings of the American Mathematical Society*, vol. 132, no. 8, pp. 2433-2439, 2004

[4] G. Faye and M. Holzer, "Asymptotic stability of the critical Fisher-KPP front using pointwise estimates," *Zeitschrift für angewandte Mathematik und Physik*, vol. 70, no. 13, 2019

[4] G. Faye and M. Holzer, "Asymptotic stability of the critical pulled front in a Lotka-Volterra competition model," *Journal of Differential Equations*, vol. 269, no. 9, pp. 6559-6601, 2020

[6] L. Lorenzi, A. Lunardi, G. Metafune, D. Pallara, "Analytic Semigroups and Reaction-Diffusion Problems," https://people.dmi.unipr.it/alessandra.lunardi/LectureNotes/I-Sem2005.pdf, 2005