Vertex Operator Algebras

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December 3, 2021

Formal Delta Function

Definition [1]

The *formal delta function* is the formal series

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n \in \mathbb{C}[[x, x^{-1}]].$$

Vertex Algebra

Definition [1]

A *vertex algebra* is a vector space V together with a linear map

$$Y(\cdot, x) : V \to (\text{End } V)[[x, x^{-1}]]$$
$$v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}$$

and distinguished vector $\mathbf{1} \in V$ called the *vacuum vector* satisfying the following conditions $\forall u, v \in V$:

- $u_n v = 0$ for n sufficiently large (*truncation condition*)
- $Y(\mathbf{1}, x) = \text{id}_V$ (*vacuum property*)
- $Y(v, x)\mathbf{1} \in V[[x]]$ and $\lim_{x \to 0} Y(v, x)\mathbf{1} = v$. (*creation property*)
- $x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2)$. (*Jacobi identity*)

Example: $V = \mathbb{C}$

Example

Define $Y(\cdot, x) : \mathbb{C} \to \mathbb{C}[[x, x^{-1}]]$ by sending $v \mapsto v = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}$ where $v_n = v$ if $n = -1$ and $v_n = 0$ if $n \neq -1$. Set the vacuum vector $\mathbf{1} = 1 \in \mathbb{C}$. Then $(\mathbb{C}, Y, \mathbf{1})$ has the structure of a vertex algebra.

Vertex Algebra Module

Definition [1]

Let $(V, Y, \mathbf{1})$ be a vertex algebra. A *V-module* is a vector space W together with a linear map

$$Y_W(\cdot, x) : V \to (\text{End } W)[[x, x^{-1}]]$$
$$v \mapsto Y_W(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}$$

satisfying the following conditions $\forall u, v \in V, w \in W$:

- $u_n w = 0$ for n sufficiently large (*truncation condition*)
- $Y_W(\mathbf{1}, x) = \text{id}_W$ (*vacuum property*)
- $x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(u, x_1) Y_W(v, x_2) -$
  $x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W(v, x_2) Y_W(u, x_1) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(Y(u, x_0)v, x_2)$
  (*Jacobi identity*)

Examples of Modules

Theorem

Let $(V, Y, \mathbf{1})$ be a vertex algebra. Define the linear map

$$Y_V(\cdot, x) : V \to (\text{End } V)[[x, x^{-1}]] \tag{1}$$

$$v \mapsto Y_V(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \equiv Y(v, x) \tag{2}$$

Then (V, Y_V) is a V-module. (This is often called the *adjoint module*.)

Theorem

Any $\mathbb{C}$-module is of the form (W, Y_W) where W is an arbitrary vector space and $Y_W(\cdot, x) : \mathbb{C} \to (\text{End } W)[[x, x^{-1}]]$ sends $u \mapsto u \, \text{id}_W$.

Vertex Operator Algebra

Definition [1]

Let $(V, Y, \mathbf{1})$ be a vertex algebra. This vertex algebra combined with a *conformal vector* $\omega \in V$ is a *vertex operator algebra* if V has the $\mathbb{Z}$-grading

$$V = \bigoplus_{n \in \mathbb{Z}} V_{(n)} \tag{3}$$

with grading restrictions $\dim V_{(n)} < \infty$ for all $n \in \mathbb{Z}$ and $V_{(n)} = 0$ for n sufficiently negative. We define the notation $\text{wt } v \equiv n$ if $v \in V_{(n)}$. We require $\mathbf{1} \in V_{(0)}$ and $\omega \in V_{(2)}$. Moreover, the following additional axioms must be satisfied for $u, v \in V$: (1) the Virasoro algebra relations

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c_V \tag{4}$$

for some $c_V \in \mathbb{C}$ called the *central charge* where we define the $L(n)$ by

$$Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}, \tag{5}$$

(2) compatibility of $L(0)$ with the grading: $L(0)v = nv = (\text{wt } v)v$ for $n \in \mathbb{Z}$ and $v \in V_{(n)}$, and (3) the $L(-1)$-derivative property

$$Y(L(-1)v, x) = \frac{d}{dx}Y(v, x). \tag{6}$$

Finite Dimensional Vertex Operator Algebras

Theorem

V is a finite dimensional commutative associative $\mathbb{C}$-algebra with identity $\mathbf{1}$ if and only if V is a finite dimensional vertex operator algebra (over $\mathbb{C}$). Moreover, the vertex operator algebra has the following structure: the conformal vector $\omega = 0$, the central charge $c_V = 0$, we have $V_{(0)} = V$ and $V_{(n)} = 0$ for $n \neq 0$, and

$$Y(u, x)v \equiv uv \tag{7}$$

for all $u, v \in V$.

Example

Let $V = \mathbb{C}[x]/(x^d)$ for $d \geq 1$. Then V is a d-dimensional vertex operator algebra where $c_V = 0$, $\omega = 0$, and the map $Y(\cdot, x) : V \to (\text{End } V)[[x, x^{-1}]]$ is defined by the usual product for the $\mathbb{C}$-algebra V: i.e. $Y(u, x)v = uv$.

Vertex Operator Algebra Module

Definition [1]

Let V be a vertex operator algebra. A V-module is a module W for V viewed as a vertex algebra such that

$$W = \bigoplus_{h \in \mathbb{C}} W_{(h)}, \tag{8}$$

where $W_{(h)} = \{w \in W | L(0)w = hw\}$, the subspace of W of vectors of weight h, and such that the grading restriction conditions $\dim W_{(h)} < \infty$ for $h \in \mathbb{C}$ and $W_{(h)} = 0$ for h whose real part is sufficiently negative.

Modules of Finite Dimensional Vertex Operator Algebras

Theorem

Let V be a finite dimensional vertex operator algebra. A vector space W is a module of the vertex operator algebra V if and only if W is finite dimensional with the $\mathbb{C}$-grading defined by $W_{(0)} = W$ and $W_{(h)} = 0$ for $h \neq 0$, $Y_W(\cdot, x)$ is constant, $[Y_W(u, x_1), Y_W(v, x_2)] = 0$ for all $u, v \in V$, $Y_W(\mathbf{1}, x) = \mathrm{id}_W$, and $Y_W(u \cdot v, x) = Y_W(u, x)Y_W(v, x)$.

References

[1] James Lepowsky and Haisheng Li. *Introduction to vertex operator algebras and their representations*. Vol. 227. Springer Science & Business Media, 2004.