Combinatorics of Cohomology Rings of the Peterson Variety

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Background: Complete Flag Variety

Definition

$$X = Fl(\mathbb{C}^n) = \{0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n | \dim(V_i) = i\}$$

The flag variety X is a space whose elements can be thought of as a chain of vector spaces. It can be represented by an $n \times n$ matrix due to the homeomorphism $Fl(\mathbb{C}^n) \cong Gl(n, \mathbb{C})/B$ where B is the subgroup of upper triangular matrices in $Gl(n, \mathbb{C})$.

Example

$$Fl(\mathbb{C}^2) = \left\{0 \subset V_1 \subset \mathbb{C}^2\right\} \cong S^2$$

Background: Peterson Variety

Definition

The Peterson variety Y is the collection of complete flags satisfying the condition $MV_i \subset V_{i+1}$ for $1 \leq i \leq n-1$ where M is a principal nilpotent operator.

Example

$$\text{Let } M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \text{ Then } \begin{pmatrix} a & b & 1 \\ b & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in Y \ \forall a, b \in \mathbb{C}$$

X has $n!$ fixed points, which can be represented by permutation matrices. Y has 2^{n-1} fixed points, which can be represented by block diagonal permutation matricies where each block is antidiagonal.

Background: Bruhat Order Diagrams

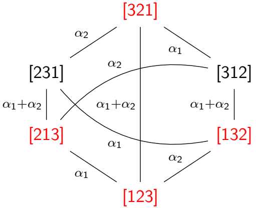

Figure: Bruhat order 3 Hasse Diagram

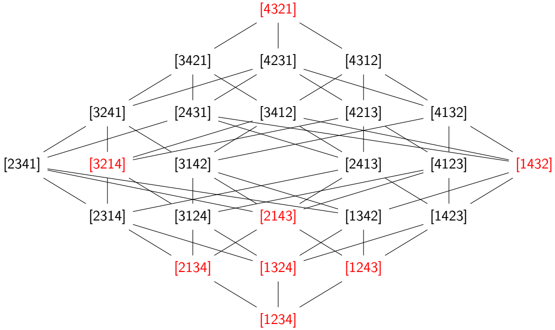

Figure: Bruhat order 4 Hasse Diagram

Research Goal

- We look at a certain variety X together with a subvariety Y and consider a circle S acting on X, under which Y is invariant.
- We associate to each space a graded ring.
- Given $Y \hookrightarrow X$, there is a natural induced surjective map $H_S^*(X) \to H_S^*(Y)$ which we want to describe.
- $H_S^*(X)$ and $H_S^*(Y)$ each have a module basis we want to explore.
- We want to find the image of the basis elements of $H_S^*(X)$ in terms of basis elements of $H_S^*(Y)$.

Equivariant Cohomology

- The equivariant cohomology $H_T^*(X)$ of a flag variety X can be regarded as a subring of

$$\bigoplus_{S_n} \mathbb{Q}[\alpha_1, ..., \alpha_{n-1}].$$

- Every equivariant cohomology class is represented by an $n!$-tuple of polynomials.

- The goal is to restrict certain classes to the Peterson variety Y.

Schubert Classes

- To a certain subvarieties called Schubert varieties X^w of X we can associate a Schubert class $\sigma_w \in H_T^*(X)$. Moreover, the σ_w's for $w \in S_n$ form a module basis for the equivariant cohomology ring.

- A subset of these classes restricted to Y form a module basis $\{p_A\}$ where $A \subseteq \{1, ..., n-1\}$ for the equivariant cohomology $H_S^*(Y)$.

Relationship Between Classes

$$\begin{array}{ccccc}
H_T^*(G/B) & \xrightarrow{\pi} & H_S^*(G/B) & \xrightarrow{\iota^*} & H_S^*(Y) \\
\downarrow & & \downarrow & & \downarrow \\
H_T^*((G/B)^T) & \longrightarrow & H_S^*((G/B)^S) & \xrightarrow{\iota^*_{fps}} & H_S^*(Y^S) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\bigoplus_{w\in W} H_T^* & \xrightarrow{\bigoplus_{w\in W}\pi} & \bigoplus_{w\in W} H_S^* & \longrightarrow & \bigoplus_{w_A\in S_n} H_S^*
\end{array}$$

Encoding Process

Once we had familiarized ourselves with the background information and computed some of these lower-order classes by hand, we moved to creating a code that would calculate these classes by hand, as well as write them as a linear combination of the basis elements

- Originally encoded in Matlab, but calculating all reduced words (permutation written as a sequence of simple reflections - not unique) of a single permutation increased computation time
- Recoded in SageMath, which has built in Permutation and Subword Complex classes, significantly reducing calculation time
- SageMath code was able to compute all classes as a linear combination of basis elements for n=6 in about 10 minutes.

Encoding Process cont.

```
def Billey(r,c):
    s=r.reduced_word_lexmin()
    d=c.reduced_word_lexmin()
    t=var('t')
    term=t^len(d)
    if len(d)==0:
        return term
    coeff=0
    iList=BilleyIndices(r,c)
    for I in iList:
        coeff_I=1
        for i in I:
            coeff_I*=BilleyCoeff(s[:i+1])
        coeff+=coeff_I
    return term*coeff
```

Figure: Sample Code

[3, 1, 2, 4]	[2, 1]	1	[[1, [1, 2]]]
[1, 4, 2, 3]	[3, 2]	1	[[1, [2, 3]]]
[4, 1, 2, 3]	[3, 2, 1]	1	[[1, [1, 2, 3]]]
[2, 4, 1, 3]	[1, 3, 2]	2	[[2, [1, 2, 3]]]
[3, 1, 4, 2]	[2, 1, 3]	2	[[2, [1, 2, 3]]]
[3, 2, 1, 4]	[1, 2, 1]	2	$[[t, [1, 2]], [1, [1, 2, 3]]]$
[1, 4, 3, 2]	[2, 3, 2]	2	$[[t, [2, 3]], [1, [1, 2, 3]]]$
[3, 4, 1, 2]	[2, 1, 3, 2]	2	$[[2t, [1, 2, 3]]]$
[4, 2, 1, 3]	[1, 3, 2, 1]	3	$[[2t, [1, 2, 3]]]$
[3, 2, 4, 1]	[1, 2, 1, 3]	3	$[[2t, [1, 2, 3]]]$
[2, 4, 3, 1]	[1, 2, 3, 2]	3	$[[2t, [1, 2, 3]]]$
[4, 1, 3, 2]	[2, 3, 2, 1]	3	$[[2t, [1, 2, 3]]]$
[4, 3, 1, 2]	[2, 1, 3, 2, 1]	5	$[[2t^2, [1, 2, 3]]]$
[3, 4, 2, 1]	[1, 2, 1, 3, 2]	5	$[[2t^2, [1, 2, 3]]]$
[4, 2, 3, 1]	[1, 2, 3, 2, 1]	6	$[[2t^2, [1, 2, 3]]]$
[4, 3, 2, 1]	[1, 2, 1, 3, 2, 1]	16	$[[2t^3, [1, 2, 3]]]$

Figure: Output for n=4

Transposition Conjecture

Let t_{ij} be the transposition of i, j, where $i < j$. Let $m := j - i$. Then

$$i^*(\sigma_{t_{ij}}) = \sum_{k=0}^{m-1} \sum_{h=0}^{k} h! \binom{k}{h}^2 \binom{m-1}{k}^2 t^h p_{\{1+i+k-m, 2+i+k-m, \ldots, j+k-h-1\}}$$

excluding any terms where $1 + i - k - m < 1$ or $j + k - h \geq n$.

$$i^*(\sigma_{[1,2,3,4,10,6,7,8,9,5,11,12,13,14]}) = 24t^4 p_{\{5,6,7,8,9\}} + 96t^3 p_{\{4,5,6,7,8,9\}} + 96t^3 p_{\{5,6,7,8,9,10\}}$$
$$+72t^2 p_{\{3,4,5,6,7,8,9\}} + 288t^2 p_{\{4,5,6,7,8,9,10\}} + 72t^2 p_{\{5,6,7,8,9,10,11\}}$$
$$+16tp_{\{2,3,4,5,6,7,8,9\}} + 144tp_{\{3,4,5,6,7,8,9,10\}} + 144tp_{\{4,5,6,7,8,9,10,11\}}$$
$$+16tp_{\{5,6,7,8,9,10,11,12\}} + p_{\{1,2,3,4,5,6,7,8,9\}} + 16p_{\{2,3,4,5,6,7,8,9,10\}}$$
$$+36p_{\{3,4,5,6,7,8,9,10,11\}} + 16p_{\{4,5,6,7,8,9,10,11,12\}} + p_{\{5,6,7,8,9,10,11,12,13\}}$$

When $i \geq m, j \leq n - m$, the coefficients for the Peterson classes of $i^*(\sigma_{t_{ij}})$ can be listed as a triangle:

$m = 4$

t^4					24				
t^3				96		96			
t^2			72		288		72		
t^1		16		144		144		16	
t^0	1		16		36		16		1

$m = 5$

t^5						120					
t^4					600		600				
t^3				600		2400		600			
t^2			200		1800		1800		200		
t^1		25		400		900		400		25	
t^0	1		25		100		100		25		1

Results for restriction to the long word

- **Observation:** Let ϕ be the map that sends $\alpha_i \mapsto t$. Let $\omega = s_{i_1} \ldots s_{i_m}$ where every simple reflection s_{i_j} commutes with every other simple reflection in the word ω. Let ν be the long word. Then

$$\phi([\sigma_\omega]^\nu) = t^m \prod_{k=1}^{m} i_k(n - i_k).$$

- **Observation:** In the notation of the previous observation,

$$\phi([\sigma_\nu]^\nu) = \prod_{k=1}^{n-1} k!t^k.$$

Conjecture for a restriction formula

- **Conjecture:** Let ϕ be the map that sends $\alpha_i \mapsto t$. Let $\omega = s_{i_1}...s_{i_m}$. Define the word $\omega' = \omega s_{i_{m+1}}$. If $s_{i_{m+1}}$ commutes with every subword of ω with length 1, then

$$\phi([\sigma_{\omega'}]^{\nu}) = t\phi([\sigma_{\omega}]^{\nu}).$$

Moving Forward

- Use code to see how conjectures hold in higher dimensions without having to calculate classes by hand
- Formalize current conjectures and create new ones based on observations
- Investigate geometric implications of our findings - what is the relationship between the corresponding flags and their basis composition?
- Create summary of background information/findings for future researchers

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