# Mathematical Exploration of New Ideas Surrounding Capillarity Understanding in Science (M.E.N.I.S.C.U.S.)

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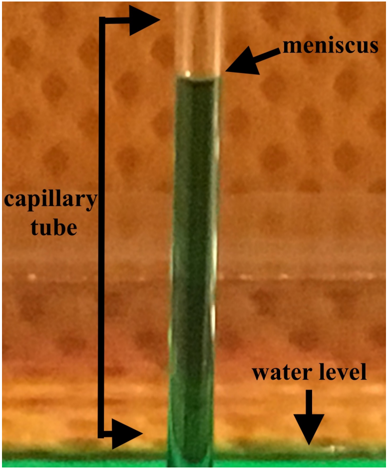
December 04, 2020

# Overview

- Introduction to Capillary Rise
- Capillary Statics
  - Jurin's Law for Tubes
  - Complex Geometries
- Capillary Dynamics
  - Navier-Stokes Equations
  - Washburn's Solution and Improvements
- Classical Results and Experiments
- Future Directions

# What is Capillary Action?

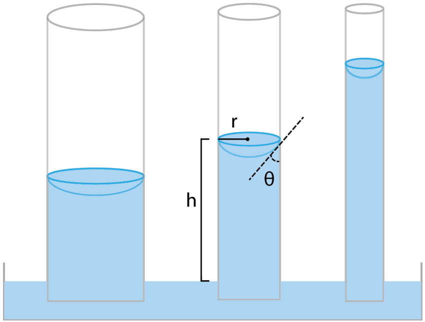
- Fluid molecules stick to their container and to each other.
- This creates **surface tension**.
- Fluid rises through its container.
- Can happen in a **tube** or a **porous material**.
- Can express the force as **capillary pressure** instead.



## Jurin's Law for Tubes (circa 1718)

$$h_{eq} = \frac{2\gamma\cos(\theta)}{\rho g r} = \frac{P_c}{\rho g}$$

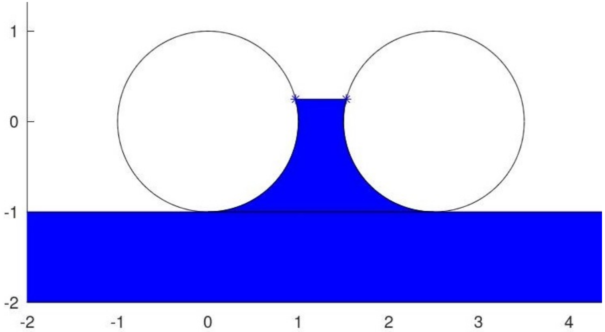
- $h_{eq}$ : equilibrium height of the fluid
- $\gamma$ : surface tension coefficient
- $\theta$ : contact angle
- $\rho$ : fluid density
- $g$ : gravitational constant
- $r$ : radius of tube
- $P_c$ : capillary pressure




Obtained by solving $\sum \vec{F} = \vec{0}$

# Capillary Statics: Complex Geometries

Goal: Find expressions for the vertical force and capillary pressure in terms of saturation.




## 2D Example: Fluid Area Bounded by Circles

$$F_V(S) = \pi\sigma(2R_0 \sin(\pi S) - l_0)\sin(\pi S + \theta)$$

Going from $2D$ to $3D$, we make the transformation $S \mapsto \dfrac{S}{f(l_0, R_0)}$

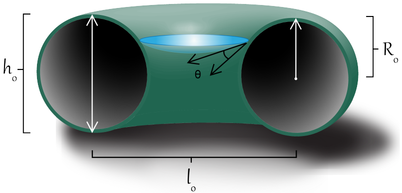


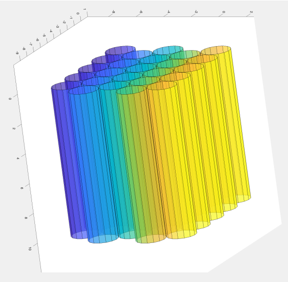

Figure 02: Fluid Bounded by Spheres

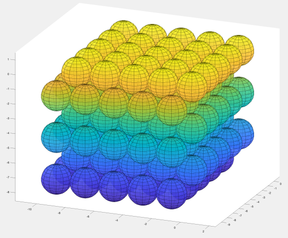
### 3D example: Volume of Fluid Bounded by Spheres

$$F_V(S) = \pi\sigma \left(2R_0 \sin\left(\frac{\pi S}{f(l_0, R_0)}\right) - l_0\right) \sin\left(\frac{\pi S}{f(l_0, R_0)} + \theta\right)$$

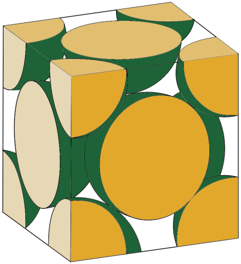
$$\mathrm{f}(\mathrm{l}_0, R_0) \approx \frac{7.7789R_0 - 3.4231}{l_0 - 1.53R_0} - 0.35066R_0 + 1.2225$$

- Washburn[4]: **tube bundle** model
- Lago and Araujo[2]: **regular sphere packing** model (more accurate)
- Use geometry to find **capillary pressure** and **equilibrium height**.
- [2] considers one packing, but there are several more.



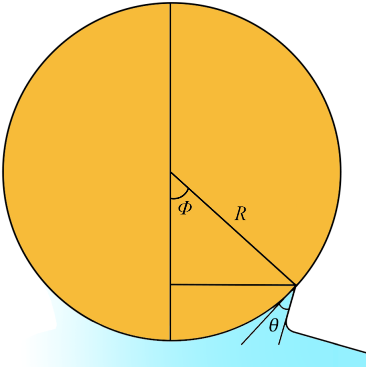


# Regular Sphere Packings

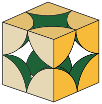
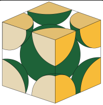
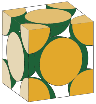
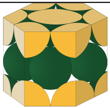


- Regular packings made of **unit cells**.
- First, calculate surface tension on a sphere.
- We can calculate $P_c(\phi)$ for a regular packing.
- $h_{eq} = \dfrac{P_c^{\min}}{\rho g}$

$$f = 2\pi R\gamma \sin\Phi \sin(\Phi + \theta)$$



# Regular Sphere Packings (Cont.)

| **Packing Type** | **Unit Cell** | **APF**[5] | $P_c$ |
|---|---|---|---|
| Simple Cubic |  | 52.36% | $\frac{\pi \sin\phi \sin(\phi+\theta)}{4-\pi\sin^2\phi} \cdot \frac{2\gamma}{R}$ |
| BCC |  | 68.02% | $\frac{\pi \sin\phi \sin(\phi+\theta)}{\frac{16}{3}-\pi\sin^2\phi} \cdot \frac{2\gamma}{R}$ |
| FCC |  | 74.05% | $\frac{\pi \sin\phi \sin(\phi+\theta)}{4-\pi\sin^2\phi} \cdot \frac{2\gamma}{R}$ |
| HCP |  | 74.05% | $\frac{\pi \sin\phi \sin(\phi+\theta)}{2\sqrt{3}-\pi\sin^2\phi} \cdot \frac{2\gamma}{R}$ |

Reminder: $P_c = \cos\theta \cdot \frac{2\gamma}{R}$ for a tube.

# Capillary Tube: Fluid Dynamics

## Navier-Stokes Equations (circa 1821)

(1)

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla P + \mu\nabla^2\vec{u} + \rho\vec{g}$$

(2)

$$\nabla\cdot\vec{u} = 0$$

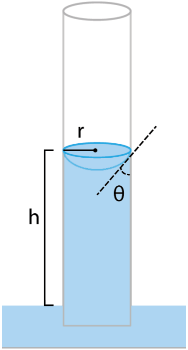
- $\vec{u}$: fluid velocity  P: fluid pressure  $\mu$: viscosity
  t: time  $\rho$: density  $\vec{g}$: gravity
- (1) is $m\vec{a} = \vec{F}$ applied to fluids.
- (2) means the fluid is incompressible.
- We can add extra restrictions and boundary conditions to model capillary action.

# Capillary Dynamics in Porous Materials

## Washburn Equation (1921)

$$h(t) = \sqrt{Dt}, \text{ where } D := \frac{\gamma r \cos\theta}{2\mu}$$

- Navier-Stokes solved for capillary tube
- Extends to bundles of tubes
- Square root function for $h(t)$ also models flow between two parallel plates.
- **Assumptions:**
  - One-dimensional flow
  - Approximately steady flow
  - Constant capillary pressure
  - Gravity is negligible.
  - No-slip condition and conservation of mass




$\gamma$: surface tension coefficient
$\mu$: viscosity

# Washburn Extended to Unsteady Flow

- Remove the assumption that the velocity is constant over time.
- The pressure gradient does not depend on the position coordinates but can depend on time.
- Special case: Assume a constant pressure gradient $\dfrac{\partial P}{\partial z} = -G$.
- More complex case: Assume $\dfrac{\partial P}{\partial z} = -\dfrac{P_C}{h(t)}$.

# Analytical Solution to Constant Pressure Gradient Case

Under a constant pressure gradient the solution is

$$
\begin{aligned}
w(r,t) &= \frac{G-\rho g}{4\mu}(R^2 - r^2) \\
&\quad - \frac{2(G-\rho g)R^2}{\mu}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^3 J_1(\lambda_n)} J_0\left(\lambda_n \frac{r}{R}\right) e^{-\lambda_n^2 \frac{\mu}{\rho R^2} t} \\
h(t) &= \frac{(G-\rho g)R}{3\mu} t \\
&\quad + \frac{2\pi\rho(G-\rho g)R^3}{\mu^2}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^5} H_0(\lambda_n)\left(e^{-\lambda_n^2 \frac{\mu}{\rho R^2} t} - 1\right)
\end{aligned}
$$

where $w$ is the velocity distribution and $h$ is the height of the fluid.

# Integro-Differential Equations for Extended Washburn

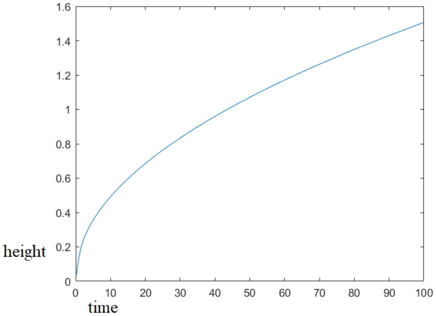
Assume $\dfrac{\partial P}{\partial z} \equiv -\dfrac{P_C}{h(t)}$.

The system is given by

$$\begin{aligned} \frac{\partial w}{\partial t} - \frac{1}{\rho}\frac{P_C}{h(t)} - \frac{\mu}{\rho}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + g &= 0 \\ \frac{dh}{dt} - \frac{2}{R^2}\int_0^R w(r,t)rdr &= 0 \\ w(R,t) &= 0 \\ w(r,0) &= 0 \\ h(0) &= 0 \end{aligned}$$

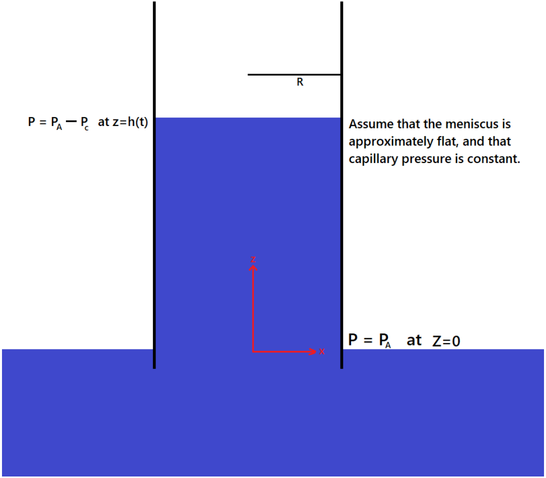
# Numerical Simulation of Unsteady Flow

- We studied an analogous problem involving the flow of a liquid confined between parallel walls.
- It is essentially a 2-dimensional problem, different from the 3-D problems of flow in tubes.
- Discretizing into a many distinct $x$-values allows us to convert the PDE into a system of many ODEs.

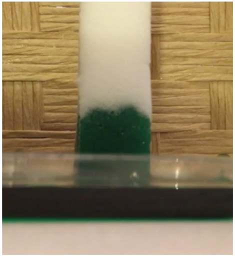


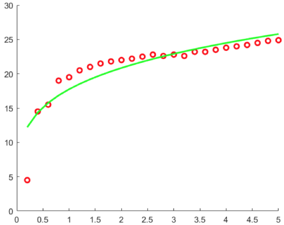
- Keep assumptions related to the Washburn-type model for parallel plates, but allow velocity to change over time.
- Assume that:

$$\frac{\partial P}{\partial z} = -\frac{P_C}{h(t)}$$



# Experiments: Unsteady Flow

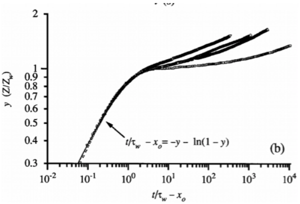




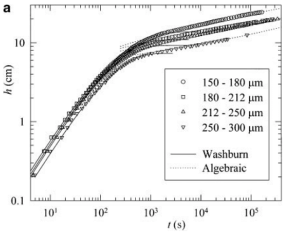

Unsteady Flow in a Magic Eraser

- 1% food coloring in DI water solution
- Fluid reservoirs at different heights joined by plastic tubing
- In general: $h \propto t^p$
- Power law least-squares approximation: $p \approx 0.2322$

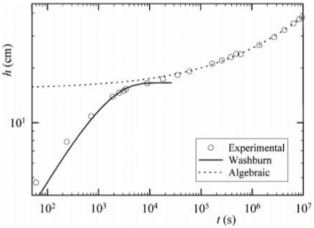
# Experiments: Classical Results



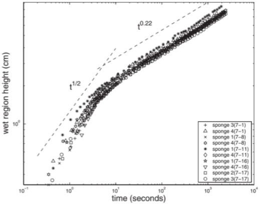

Delker, Pengra and Wong (1996)
Water, Glass Column, Glass Beads




Lago and Araujo (2000)
Water, Glass Column, Glass Beads

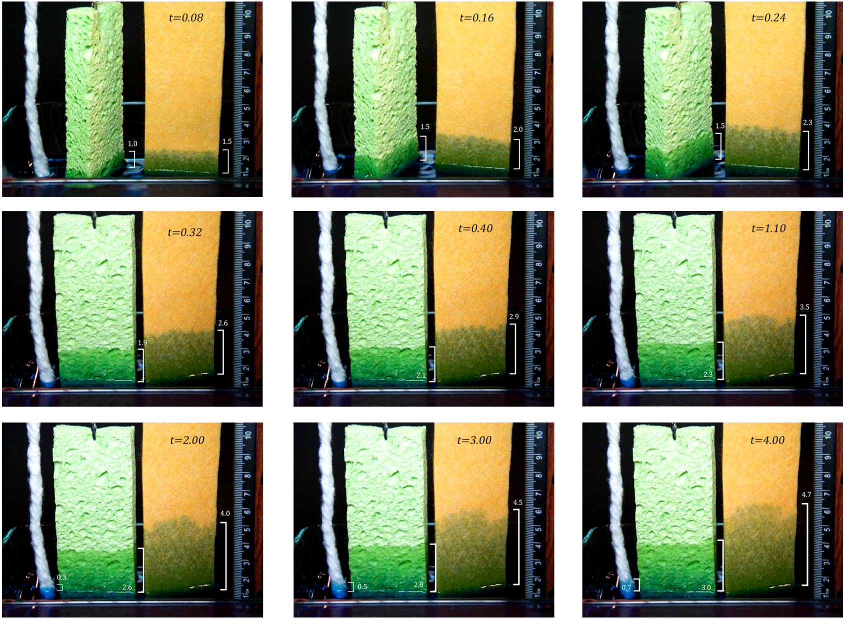



Lago and Araujo (2000)
Oil, Glass Column, Sand Packing



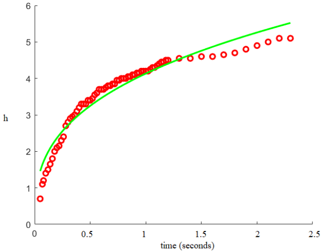

Siddique, Anderson, and Bondarev (2009)
Water, Sponge

# Experiments: Different Porous Materials

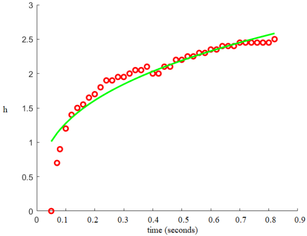


# Experiments: Different Porous Materials

Method used: tap water in stationary fluid reservoir



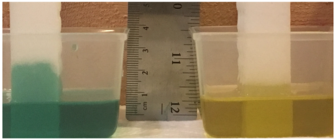

Shammy
$p \approx 0.3475$



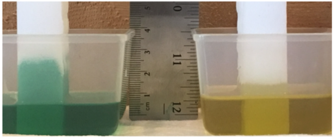

Green Sponge
$p \approx 0.3354$

Method used: 1% food coloring in DI water, oil in stationary fluid reservoir

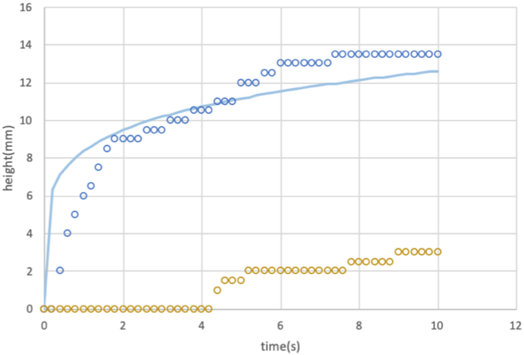
$H_{eq(oil)} \approx 7.5mm$ $\quad H_{eq(water)} \approx 24mm$ $\quad p_{water} \approx 0.1772$



15 minutes (Initial Equilibrium)

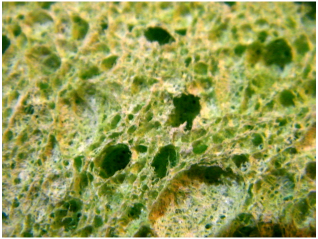


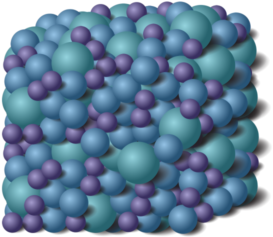
8 hours (Overnight Equilibrium)



# Ideas Moving Forward

- **Mixture Theory:** How liquids *and* gases permeate a material
- **Deformable Materials:** How material expansion and deformation affect capillarity
- **More Simulations:** Numerically solve the problem of unsteady flow in a tube, like we did for parallel plates
- **Random Sphere Packings:** A better approximation for porous materials





# Acknowledgements

We would like to thank our project mentor, Dr. Anderson. We would also like to thank Dr. Siddique, who joins us from Penn State; Jessica Masterson, our Graduate Research Assistant; and MEGL for supporting our research.

# References


T. Delker, D. Pengra, and P. Wong.
Interface pinning and the dynamics of capillary rise in porous media.
*Physical Review Letters*, 76(16):2902–2905, 1996.

M. Lago and M. Araujo.
Capillary Rise in Porous Media.
*Journal of Colloid and Interface Science*, 234(1):35–43, Feb. 2001.

J. I. Siddique, D. M. Anderson, and A. Bondarev.
Capillary rise of a liquid into a deformable porous material.
*Physics of Fluids*, 21(1):013106, 2009.

E. Washburn.
The dynamics of capillary flow.
*Physical Review*, 17(3):273–283, 1921.

Wikipedia contributors.
Atomic packing factor — Wikipedia, the free encyclopedia, 2020.
[Online; accessed 4-December-2020].

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