Cores and Hulls of Ideals of Commutative Rings

Dr. Rebecca R.G., Matthew South, Joseph Bidinger, Ethan Clelland, Aidan Donahue

George Mason University, MEGL

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Rings

Definition

A **ring** is a set R equipped with two operations "addition" $+$ and "multiplication" $\cdot$ where:

- R is closed under addition
- R has an additive identity 0_R
- R contains additive inverses for all $r \in R$
- $+$ is associative and commutative
- R is closed under multiplication
- $\cdot$ is associative (and has identity 1)
- $\cdot$ distributes over $+$

If multiplication is commutative, R is a commutative ring.

Examples of Commutative Rings

- Number Rings: $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{R}$
- Power Series Rings: $\mathbb{Z}[[t]]$ - Power series with integer coefficients.

Ideals of Commutative Rings

Definition

A subset I of a commutative ring R is an **ideal** of R if:

- $0_R \in I$
- I is closed under same addition as R
- Every element in I has an additive inverse contained in I
- $r \cdot i \in I \; \forall \; i \in I$ and $r \in R$

Examples

- $2\mathbb{Z}$ – the set of even integers is an ideal of $\mathbb{Z}$. If you multiply any non-even integer by an even one, you obtain an even integer.
- The set of all power series with a factor of t in $\mathbb{Z}[[t]]$ is an ideal. Example: $t(t+1) \in I$, $t^2+1 \notin I$, $t^4+t^3+t^2+t$ has a factor of t.

Finitely Generated Ideals

Definition

The set of all elements in $r \in R$ such that $r = a \cdot b$ for some $b \in R$ is called the ideal generated by a and is written (a). Given n ideals $(a_1), (a_2), \ldots, (a_n)$, $(a_1, a_2, \ldots, a_n)$ is the smallest ideal containing all (a_i).

Example

- $(0) = \{0\}$ is an ideal of any ring and is contained in all other ideals
- The set of power series in $\mathbb{Z}[[t]]$ with a factor of t^2 is the ideal (t^2). The ideal (2) is the set comprised of power series with even coefficients.
- $(t^2, 2) = \{\alpha + a_1 t^2 + a_2 t^3 + \ldots \mid \alpha \in 2\mathbb{Z}, a_i \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z}[[t]]$. Notice the ideal $\{a_0 + a_1 t^2 + a_2 t^3 + \ldots \mid a_i \in \mathbb{Z}\}$ *does* contain both (t^2) and (2), but properly contains the previous set.

Numerical Semigroup Rings

Definition

A **field** is a ring k where every nonzero element has a multiplicative inverse contained in k.

Definition

Let k be an infinite field with characteristic $p > 0$. A **numerical semigroup ring** is a power series ring $k[[t^{s_1}, t^{s_2}, \dots, t^{s_k}]]$ whose elements are k-linear combinations of monomials of the generators. We choose $s_i \in \mathbb{N}$ and $\gcd(s_1, s_2, \dots, s_k) = 1$.

Example

- $k[[t]] = \{a_0 + a_1 t + a_2 t^2 + \dots\}$ contains t and t^3 as elements
- $k[[t^2, t^5]] = \{a_0 + a_1 t^2 + a_2 t^5 + a_3 t^6 \dots\}$ does NOT contain t and t^3.

Why are ideals in these rings interesting?

Ideal	$k[[t]]$	$k[[t^2, t^5]]$
(t^3)	Exists	Does not exist
(t^2)	$\{a_0 t^2 + a_1 t^3 + \dots\}$	$\{a_0 t^2 + a_1 t^4 + a_2 t^6 + a_3 t^7 \dots\}$
$(t^5 + t^8)$	$\{a_0 t^5 + a_1 t^6 + \dots\}$	$\{a_0(t^5 + t^8) + a_1 t^7 + a_2 t^9 + a_3 t^{10} \dots\}$

Ideals with the same generator can be drastically different depending on the ring

Interior Operations

Definition

Let I and J be ideals of a ring R. An operation *int* : Ideals of R $\rightarrow$ Ideals of R is called an **interior operation** if:

- $I_{int} \subseteq I$
- $(I_{int})_{int} = I_{int}$
- For $I \subseteq J$, $I_{int} \subseteq J_{int}$

Example

The mapping that takes any ideal in R to the zero ideal is an interior operation.

$$I_{int} = (0)$$

The interior operation we are focused on is the tight interior. The tight interior of an ideal is denoted by I_*

Tight Interior

Theorem (Vassilev 2021)

Let $k[[t^{a_1}, t^{a_2}, \ldots, t^{a_i}]]$ be a numerical semigroup ring. Then

$$I_* = (t^{c_I}, t^{c_I+1}, ..., t^{c_I+m-1})$$

where $c_I = \min\{n \in \mathbb{N} : t^m \in I \ \forall \ m \geq n\}$.

Example

In the ring $k[[t^2, t^5]]$, we have that $(t^4, t^7)_* = (t^6, t^7)$.

*-Hull

Definition

Let I_S be the set of ideals J with $I \subseteq J, J_* = I_*$ Then,

$$* - hull(I_S) = \sum_{J_i \in I_S} J_i$$

We say that all J_i are ***-expansions** of I.

Example

In the ring $k[[t^2, t^5]]$, we have that , for all $a \in k$, $(t^2 + at^5)$ is a maximal *-expansion of (t^4, t^7), which has tight interior (t^6, t^7). Since $(t^2 + at^5)_* = (t^4, t^7)_*$, we see that $* - hull(t^4, t^7) = \sum_{a \in k}(t^2 + at^5) = (t^2, t^5)$.

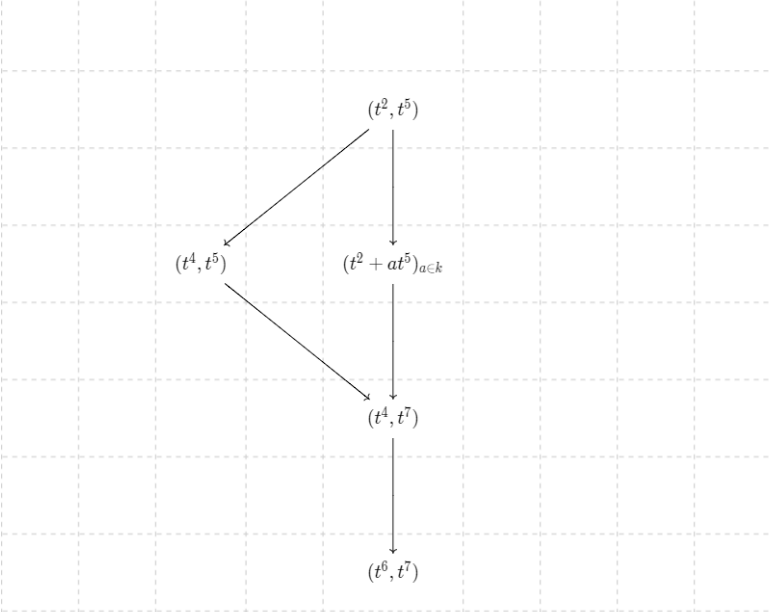
Motivation for tight-interior and *-hulls

Tight closure is an operation related to the singularities of a ring, and so is a topic of active interest in commutative algebra.

The core of an ideal was originally defined using integral closure, which is related to the results of the Briancon-Skoda Theorem, and other results in commutative algebra. However, we can define a core for any closure operation, such as for tight-closure.

Tight interior is a dual to tight closure [ES14], and *-hulls are a new tool which we would like to understand in order to further our knowledge of tight interior, tight closure, and the singularities of commutative rings.

Visualizing ideals in $k[[t^2, t^5]]$



Examples of Tight Interiors

Example

In $k[[t^2, t^5]]$,

$$(t^4, t^7) = \{a_1 t^4 + a_2 t^6 + a_3 t^7 + \dots\} \implies I_* = (t^6, t^7)$$
$$(t^6, t^7) = \{a_1 t^6 + a_2 t^7 + \dots\} \implies I_* = (t^6, t^7)$$
$$(t^2 + at^5) = \{a_1(t^2 + at^5) + a_2 t^6 + a_3 t^7 + \dots\} \implies I_* = (t^6, t^7).$$

In these examples, $c_I = 6$, and $m = 2$. Also, note that $(t^2 + at^5)$ is a maximal ideal with (t^6, t^7) as it's tight interior.

Now, we can look at some examples of tight interiors for rings of higher multiplicity.

Example

In $k[[t^3, t^5, t^7]]$, for $n \geq 5$,

$$(t^n + at^{n+1} + bt^{n+2})_* = (t^{n+5}, t^{n+6}, t^{n+7}), \forall a, b \in k$$
$$(t^n, t^{n+2})_* = (t^{n+5}, t^{n+6}, t^{n+7})$$
$$(t^n, t^{n+2}, t^{n+4})_* = (t^{n+2}, t^{n+3}, t^{n+4})$$

Results Concerning a generic case

For any $n \in \mathbb{N}$, we have computed a classification of the tight interiors and $*-$*hulls* in rings of the form $k[[t^2, t^{2n+1}]]$. Let $j \geq 2n$, $1 \leq \ell \leq 2n-1$, with ℓ odd, and $a \in k$. Then

Tight Interiors

$$(t^j, t^{j+\ell})_* = (t^{j+\ell-1}, t^{j+\ell}),$$
$$(t^j + at^{j+\ell})_* = (t^{j+2n}, t^{j+2n+1}).$$

*-hulls

Let p even with $2 \leq p \leq 2n$. Then

$$* - hull(\langle t^{j+p}, t^{j+c+1} \rangle) = \sum_{a \in k} \sum_{\substack{l \\ p \geq c-l+1}} \langle t^j + at^{j+l} \rangle = \langle t^j, t^{j+c+1-p} \rangle$$

$$* - hull(\langle t^j + at^{j+l} \rangle) = \langle t^j + at^{j+l} \rangle$$

When Tight Closure coincides with *-hulls.

Theorem (Vassilev 2021)

*Let $R = k[[t^S]]$ where S is a numerical semigroup with k an infinite field of characteristic $p > 0$. If I is a tightly open ideal and J is a maximal *-expansion of I, then *-hull(I) $= J^*$.*

It turns out that we need the assumption that I is tightly open; i.e. that $I = I_*$.

The Theorem cannot be weakened.

Consider the ideals $(t^6 + at^9)$ and (t^8, t^{11}) in $k[[t^2, t^5]]$. We note that for each $a \in k$, $(t^6 + at^9)$ is a maximal *-expansion of (t^8, t^{11}). We may then compute that:

$$(t^6 + at^9)^* = (t^6, t^7), \text{ while } * - hull((t^8, t^{11})) = (t^6, t^9).$$

Note that $(t^8, t^{11}) \neq (t^8, t^{11})_* = (t^{10}, t^{11})$.

Future Goals

- Automate computations of tight interiors and $* - hulls$ using GAP, a computer algebra system.
- Generalize findings to other classes of numerical semigroup rings.

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