Cores and Hulls of Ideals of Commutative Rings

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*-hull Motivation

The core of an ideal was originally defined using integral closure by Sally and Rees in 1988. Fouli and Vassilev extended the idea of the core of an ideal to other closure operations such as the tight closure which was discovered by Huneke and Hochster. The tight closure operation and integral closure cores have been well-studied with tight closure leading to the Briancon-Skoda Theorem.

Tight interior I_* and *-hull of an ideal are a dual to the tight closure I^* and *-cores [ES14] of the ideal. The *-hull of an ideal is a new tool which we would like to understand in order to further our knowledge of tight interior, tight closure, and the singularities of commutative rings.

Ideals of Commutative Rings

A subset I of a commutative ring R is an **ideal** of R if:

- $0_R \in I$
- I is closed under same addition as R
- Every element in I has an additive inverse contained in I
- $r \cdot i \in I \ \forall\, i \in I$ and $r \in R$

Finitely Generated Ideals

The set of all elements in $r \in R$ such that $r = a \cdot b$ for some $b \in R$ is called the ideal generated by a and is written (a). Given n ideals $(a_1), (a_2), \ldots, (a_n)$, $(a_1, a_2, \ldots, a_n)$ is the smallest ideal containing all (a_i).

Interior Operations

Let I and J be ideals of a ring R. An operation int : Ideals of R $\to$ Ideals of R is called an **interior operation** if:

- $I_{int} \subseteq I$
- $(I_{int})_{int} = I_{int}$
- For $I \subseteq J$, $I_{int} \subseteq J_{int}$

Our research focuses on the tight interior operation, or I_*.

*-hulls

Let I_S be the set of ideals J with $I \subseteq J$, $J_* = I_*$. Then,

$$* - hull(I_S) = \sum_{J_i \in I_S} J_i$$

We say that all J_i are ***-expansions** of I. We can define int-hulls for other interior operations, but we only focused on the *-hull this semester.

Numerical Semigroups

Definition

Let k be an infinite field with characteristic $p > 0$. A **numerical semigroup ring** is a power series ring

$$k[[t^{s_1}, t^{s_2}, \ldots, t^{s_k}]]$$

whose elements are k-linear combinations of monomials of the generators. We choose $s_i \in \mathbb{N}$ and $\gcd(s_1, s_2, \ldots, s_k) = 1$.

Numerical semigroup rings are of particular research interest due to the ease at which we can create a lattice of ideals. These diagrams are useful for computing tight interiors and *-hulls, as it helps to illustrate which ideals contain which.

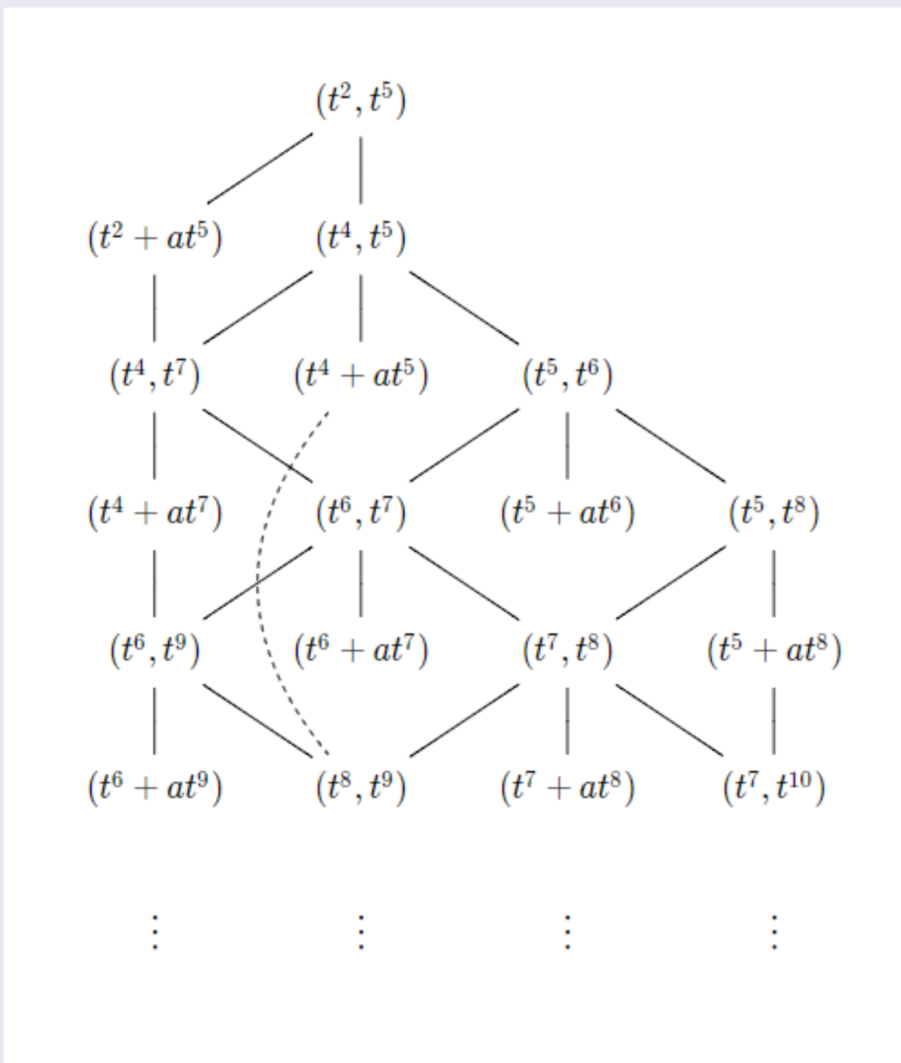
Theorem (Vassilev 2021)

Let $k[t^{a_1}, t^{a_2}, \ldots, t^{a_l}]$ with $a_1 < a_2 < \cdots < a_n$ be a numerical semigroup ring. Then

$$I_* = (t^{c_I}, t^{c_I+1}, \ldots, t^{c_I+m-1})$$

where $c_I = \min\{n \in \mathbb{N} : t^m \in I \ \forall\, m \geq n\}$ and $m = a_1$ – the lowest generating power of t.

Lattice of ideals for $k[[t^2, t^5]]$



Tight interiors in $k[[t^2, t^5]]$

Let $n \geq 4$. We have 4 classes of distinct ideals

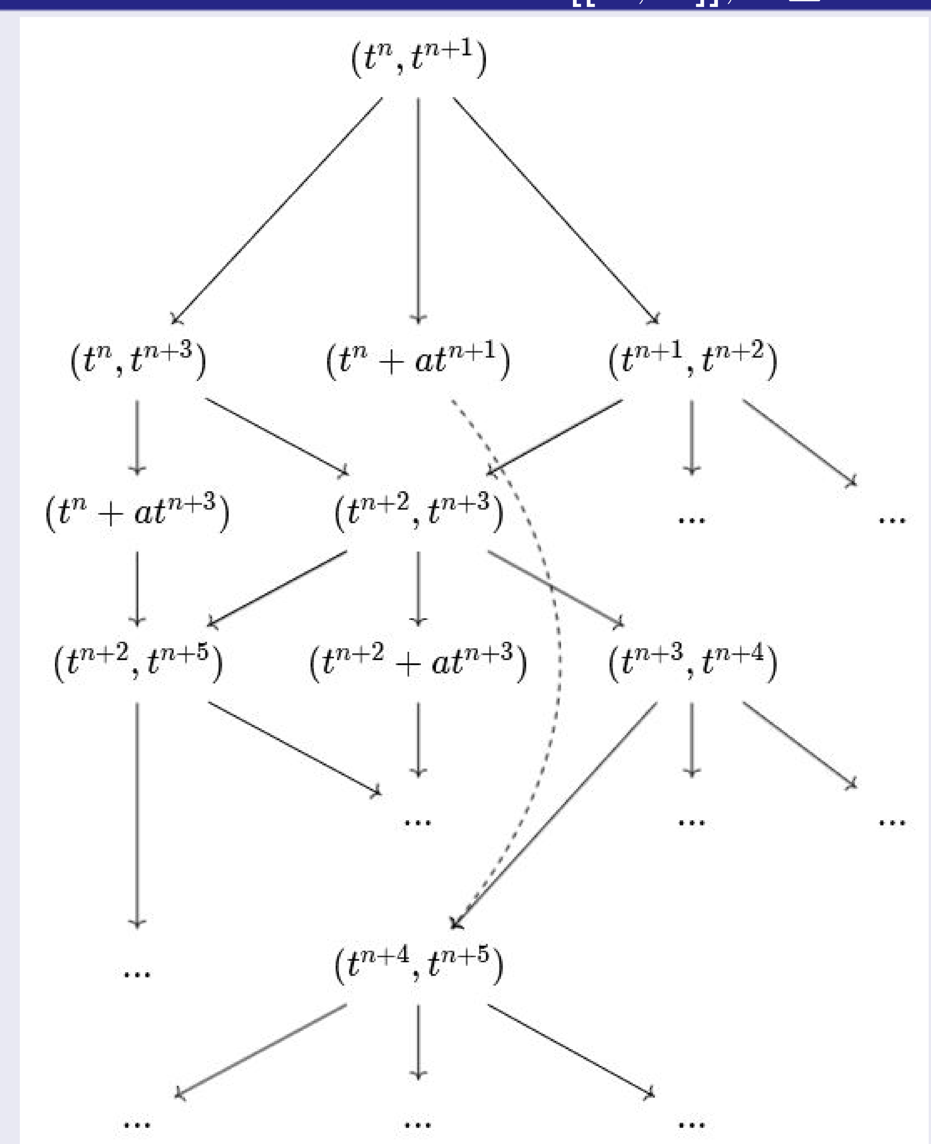
$$\langle t^n, t^{n+1} \rangle_* = \langle t^n, t^{n+1} \rangle$$
$$\langle t^n, t^{n+3} \rangle_* = \langle t^{n+2}, t^{n+3} \rangle$$
$$\langle t^n + at^{n+1} \rangle_* = \langle t^{n+4}, t^{n+5} \rangle$$
$$\langle t^n + at^{n+3} \rangle_* = \langle t^{n+4}, t^{n+5} \rangle.$$

Tight interiors in $k[[t^3, t^5, t^7]]$

For $n \geq 6$ we have many different classes of ideals.

- $(t^j + at^{j+1} + bt^{j+2}) \implies I_* = (t^{j+5}, t^{j+6}, t^{j+7})$
- $(t^j, t^{j+2}, t^{j+3}) \implies I_* = (t^{j+2}, t^{j+3}, t^{j+4})$
- $(t^j, t^{j+1}, t^{j+2}) \implies I_* = (t^j, t^{j+1}, t^{j+2})$

Generalized lattice of ideals for $k[[t^2, t^5]]$, $n \geq 4$



Example *-hull in $k[[t^2, t^5]]$

The ideals $(t^2 + at^5)$ are the maximal *-expansions of (t^4, t^7), so

$$* - hull(t^4, t^7) = \sum_{a \in k} (t^2 + at^5) = (t^2, t^5).$$

Example *-hull in $k[[t^3, t^5, t^7]]$

The ideals $(t^j + at^{j+1} + bt^{j+2})$ are maximal *-expansions of $(t^{j+5}, t^{j+6}, t^{j+7})$, so

$$* - hull(t^{j+5}, t^{j+6}, t^{j+7}) = \sum_{a,b \in k} (t^j + at^{j+1} + bt^{j+2}) = (t^j, t^{j+1}, t^{j+2})$$

The General Multiplicity 2 Case - $k[[t^2, t^{2n+1}]]$

Let $j \geq c$, $\ell \in H(2, 2n+1)$, $a \in k$, and p an even integer with $2 \leq p \leq c$

$$* - hull(t^{j+p}, t^{j+c+1}) = (t^j, t^{j+c+1-p})$$
$$* - hull(t^j + at^{j+\ell}) = (t^j + at^{j+\ell})$$

We conjecture that there are two more possible classes of *-hulls for general rings of this form. This would provide a good starting point for future research.

Future Exploration

- Create and implement algorithms using GAP, a language for commutative algebra, to assist in computing more examples of tight interiors and *-hulls. GAP has a package which provides resources for working with numerical semigroup rings, which would significantly reduce the amount of manual calculations required for these computations.
- Generalize our results to a wider variety of numerical semigroup rings, including rings with higher multiplicity.

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