Combinatorial Formulas for the Equivariant Cohomology of Peterson Varieties

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INTRODUCTION AND MOTIVATION

- We look at a variety X and a subvariety Y and consider a circle S acting on X, under which Y is invariant.
- We associate to each space a graded ring.
- Given $Y \overset{i}{\hookrightarrow} X$, there is a natural induced surjective map $i^* : H_S^*(X) \to H_S^*(Y)$ which we want to describe.
- $H_S^*(X)$ and $H_S^*(Y)$ each have a module basis we want to explore.

PROJECT GOAL

The goal is to express the restriction of Schubert classes to the Peterson variety as a linear combination of Peterson classes.

TERMINOLOGY

Complete Flag Variety

$$X = Fl(\mathbb{C}^n) = \{0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \mid \dim(V_i) = i\}$$

The flag variety X is a space whose elements can be thought of as a chain of vector spaces. It can be represented by an $n \times n$ matrix due to the homeomorphism $Fl(\mathbb{C}^n) \cong Gl(n,\mathbb{C})/B$ where B is the subgroup of upper triangular matrices in $Gl(n,\mathbb{C})$.

Peterson Variety

The Peterson variety Y is the collection of complete flags satisfying the condition $MV_i \subset V_{i+1}$ for $1 \leq i \leq n-1$ where M is a principal nilpotent operator.

Equivariant Cohomology

The equivariant cohomology $H_T^*(X)$ of a flag variety X can be regarded as a subring of

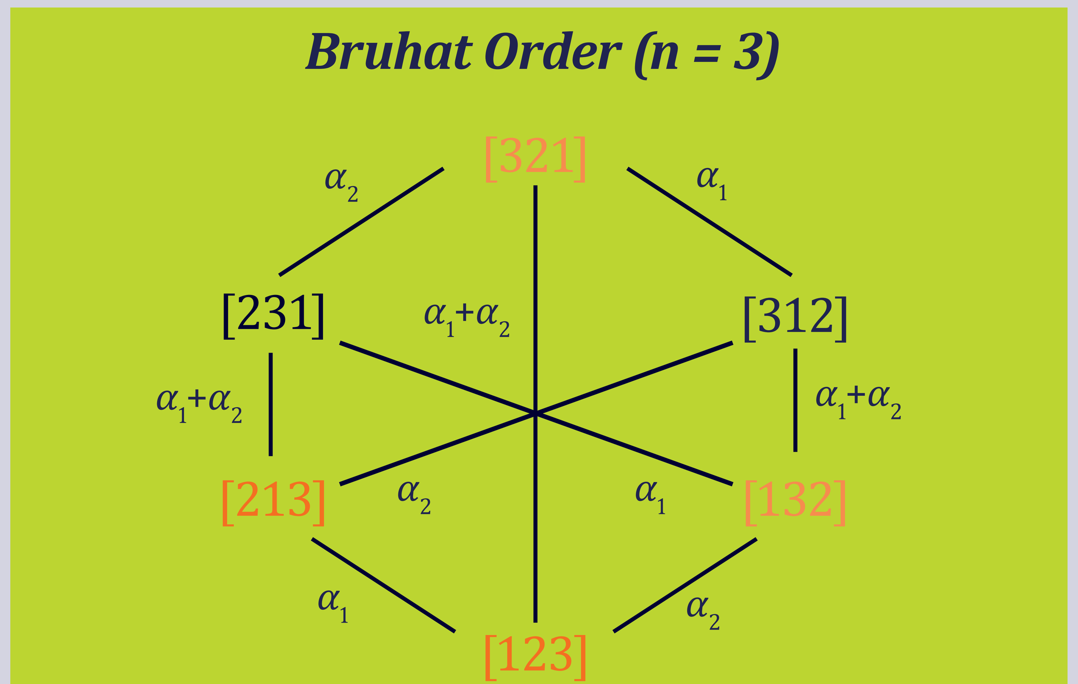
$$\bigoplus_{S_n} \mathbb{Q}[\alpha_1, \ldots, \alpha_{n-1}].$$

Every equivariant cohomology class is then represented by an $n!$-tuple of polynomials.

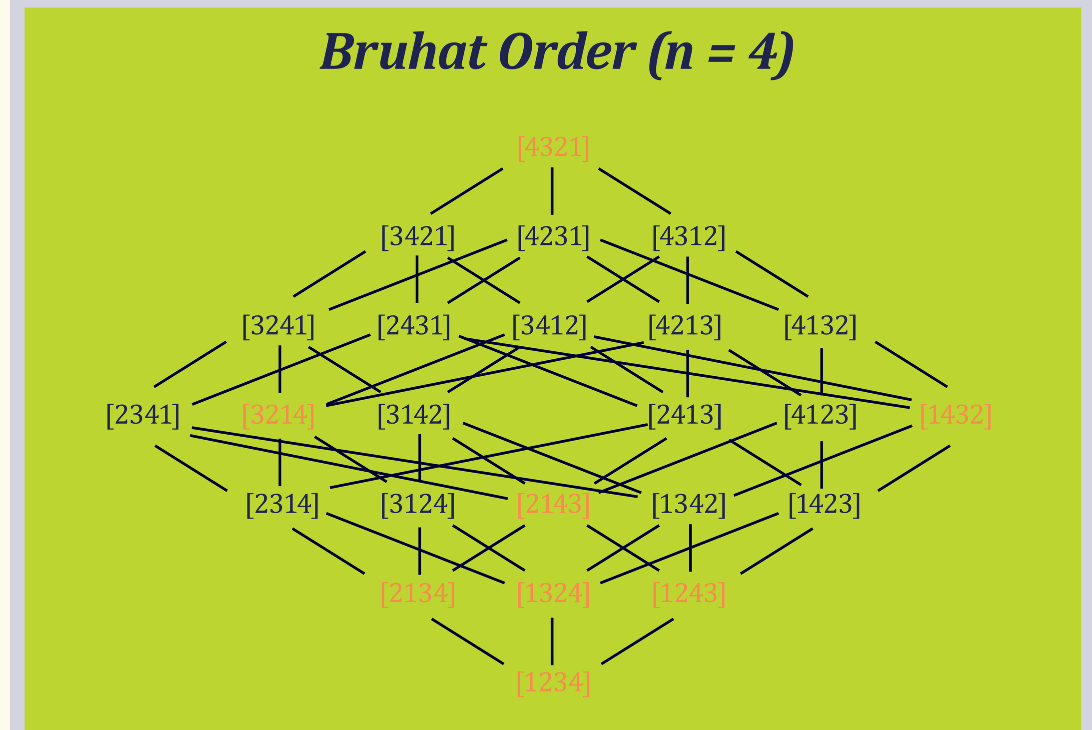
Basis for $H_T^*(X)$ and $H_S^*(Y)$

To each $v \in S_n$, we can associate a Schubert class σ_v in $H_T^*(X)$. v can be written as a permutation in one-line notation, which can in turn be represented by a product of simple reflections s_i where s_i transposes the elements in the i^{th} and $(i+1)^{\text{th}}$ position. Peterson classes p_I are Schubert classes where $v \in C$, where C is the set of Coxeter elements, which are all elements of X that can be written as a reduced word of strictly ascending simple reflections where no simple reflection is used more than once.

FIXED POINTS DIAGRAMS



X has n! fixed points, which can be represented by n × n permutation matrices. Y has 2^{n-1} fixed points, which can be represented by block diagonal permutation matrices where each block is antidiagonal.



The fixed points of Y are colored orange. The number of fixed points grows by n! for X and 2^n for Y.

BILLEY'S FORMULA

Let w,v be elements of the Weyl group and fix a reduced word $b_1 b_2 \cdots b_m$ for w. Billey's Formula for the polynomial $\sigma_v(w)$ is

$$\sigma_v(w) = \sum r_{j_1} r_{j_2} \cdots r_{j_k}$$

where the sum is over all reduced subwords $b_{j_1} b_{j_2} \cdots b_{j_k}$ for v in w, and for each j, $r_j = b_1 b_2 \cdots b_{j-1} \alpha_j$ [3]. We began by implementing the formula in MatLab and later moved to Sage, which is like Python with many built in mathematics packages. We use the map $H_T^*(X) \to H_S^*(Y)$ to send α_i to t for each i, after which $\sigma_v(w)$ becomes a monomial in t.

```
def Billey(r,c):
    s=r.reduced_word_lexmin()
    d=c.reduced_word_lexmin()
    t=var('t')
    term=t^len(d)
    if len(d)==0:
        return term
    coeff=0
    iList=BilleyIndices(r,c)
    for I in iList:
        coeff_I=1
        for i in I:
            coeff_I*=BilleyCoeff(s[:i+1])
        coeff+=coeff_I
    return term*coeff
```

Billey's Formula implemented in Sage.

PULLBACK LINEAR COMBINATIONS AND THE PETERSON CLASS BASIS

We are interested in restricting elements of X to Y to obtain the Schubert classes corresponding to each element. We can then find the pullback $i^*(\sigma_v)$ for each element $v \in X$ and write the result as a linear combination of Peterson classes, $i^*(\sigma_c)$ for each $c \in C$.

Sample calculations:

$$\sigma_{[1324]}([1432]) = \alpha_2 + \alpha_3 = 2t$$

$v = [1324]$ can only be written as s_2, and we fix $s_3 s_2 s_3$ for $w = [1432]$. Then Billey's Formula gives $\sigma_v(w) = r_2 = b_1\alpha_2 = s_3(x_3 - x_2) = x_4 - x_2 = x_4 - x_3 + x_3 - x_2 = \alpha_2 + \alpha_3$.

$i^*(\sigma_{[2143]}) = p_{\{1,3\}}$ since [2143] can be written as $s_1 s_3$ and thus is in C.

$i^*(\sigma_{[15432]}) = 2t^3 p_{\{2,3,4\}} + 10t^2 p_{\{1,2,3,4\}}$

Equivariant Cohomology Commutative Diagram

$$\begin{tikzcd}
H_T^*(G/B) \arrow[r, "\pi"] \arrow[d] & H_s^*(G/B) \arrow[r, "t^*"] \arrow[d] & H_s^*(Y) \arrow[d] \\
H_T^*((G/B)^T) \arrow[r] \arrow[d, equal] & H_s^*((G/B)^S) \arrow[r, "t^*_{fps}"] \arrow[d, equal] & H_s^*(Y^S) \arrow[d, equal] \\
\bigoplus_{w\in W} H_T^* \arrow[r, "\bigoplus_{w\in W}\pi"] & \bigoplus_{w\in W} H_s^* \arrow[r] & \bigoplus_{w_A\in S_n} H_s^*
\end{tikzcd}$$

PROGRAMMING

Sage has many functions that make it faster for smaller values of n. For larger values only the MatLab code appears to work, while Sage seems to run forever without output. The most computationally expensive part is computing restrictions using Billey's Formula; the longest computation we did, computing a pullback in $n = 14$, took about 15 hours in MatLab. The computing time for $i^*(\sigma_v)$ with the current algorithm is $O\left(\frac{(2n)!}{n!}\, n^2\right)$.

RESULTS

Conjecture:

Let $v_{i,j}$ be $[1, 2, \ldots, n]$ with i and j transposed, where $i < j$. Let $m = j - i$ be defined as the magnitude of the transposition, and note that $v_{i,j}$ can be written in reduced word form as $s_i s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_{i+1} s_i$. Then

$$i^*(\sigma_{v_{i,j}}) = \sum_{k=0}^{m-1} \sum_{h=0}^{k} h! \binom{k}{h}^2 \binom{m-1}{k}^2 t^h p_{\{1+i+k-m,\, 2+i+k-m, \ldots, j+k-h-1\}}$$

excluding any terms where $1 + i - k - m < 1$ or $j + k - h \geq n$.

Observation:

Let ϕ be the map that sends $\alpha_i \mapsto t$. Let $\omega = s_{i_1} \ldots s_{i_m}$ where every simple reflection s_{i_j} commutes with every other simple reflection in the word ω. Let v be the long word. Then

$$\phi(\sigma_\omega(v)) = t^m \prod_{k=1}^{m} i_k (n - i_k).$$

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