Geometry of Complex Networks

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December 10, 2018

Introduction

We are interested in studying the underlying geometry of a complex network. In our studies we aim to associate a space to this network so that it can be embedded into that space. When we embed our network into that space we seek to maintain the community structure and distances between nodes so that it is an isometric embedding.

The possible applications of these studies would allow the prediction of links if a new node were to be added to the network. This would allow us to determine if a node is placed in an area, where would it possibly be connected into the rest of the network. Another possible application would be to assist in inproving routing witin th network. If you understand the geometry of the network and know how it was embedded, you could then know where your destination node should theoretically be located and can then efficiently route to that new location.

Some Definitions

Definition (Complex Network)

A network is graph, $G = (V, E)$ with V vertices and E edges.

A complex network is a graph with non-trivial features that do not occur in simple networks such as lattices or random graphs but often occur in graphs modelling real systems.

General features of real world networks: sparse, algebraic degree distribution (i.e. hubs exist), community structure, exponential growth of neighbors.

Hyperbolicity Calculation Algorithm

Algorithm 1: Hyperbolicity

Input: G is a 2-connected graph
Input: pairs is the list of the $\binom{n}{2}$ pairs of vertices sorted by decreasing distances.
Result: δ, the hyperbolicity of G (observe that $2\delta = \mathrm{h_{diff}}$).

```
Let h_diff := 0;
for 1 ≤ i < (n 2) do
    (a, b) := pairs[i];
    for 0 ≤ j < i do
        (c, d) := pairs[j];
        h_diff := max {h_diff, δ_diff(a, b, c, d)} ;
        if d(a, b) ≤ h_diff then
            return h_diff /2
return h_diff /2
```

Global Airline Network as of 2001 – Guimera *et. al* 2005

- 3618 cities (nodes) with 14,142 connections (edges)
- Includes passenger and freight routes and air taxis
- Diameter is 17 representing travel from Wasu, Papa New Guinea to Brieze Norton Air Force Base
- Exhibits small world property – algebraic degree distribution with average shortest path between cities of 4.4
- More clustered than a random network due to spatial structure

Some More Definitions

Definition (δ-Hyperbolicity)

δ is a scalar value.

δ tells how "tree-like" a graph is.

The δ of a tree is always 0.

There are several ways to calculate the δ-hyperbolicity which all lead to brute force algorithms taking time proportional to n^4, where n is the number of vertices in the network.

Definition (Embedding)

When some object X is said to be embedded in another object Y, the embedding is given by some injective and structure-preserving map $f : X \to Y$. We will look at taking the V vertices and mapping them to points in Ω^n. An **isometric** embedding will preserve the exact distances between nodes in $f(V)$. A **similarity** embedding will have connected points close together in $f(V)$, and disconnected nodes will be far apart.

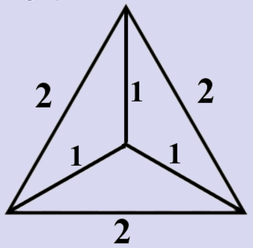
Definition (Four Point Condition)

In a graph $G = (V, E)$, given four vertices x, y, u, and $v \in V$ with $d(x, y) + d(u, v) \geq d(x, u) + d(y, v) \geq d(x, v) + d(y, u)$, the hyperbolicity of the quadruple x, y, u, v denoted as $\delta(x, y, u, v)$ is defined as:

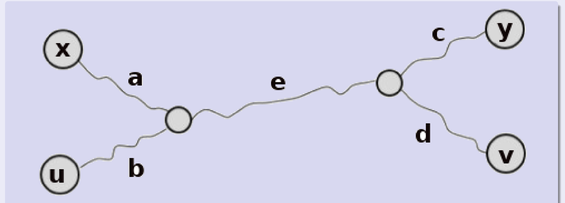
$$\delta(x, y, u, v) = \frac{d(x,y)+d(u,v)-(d(x,u)+d(y,v))}{2}$$

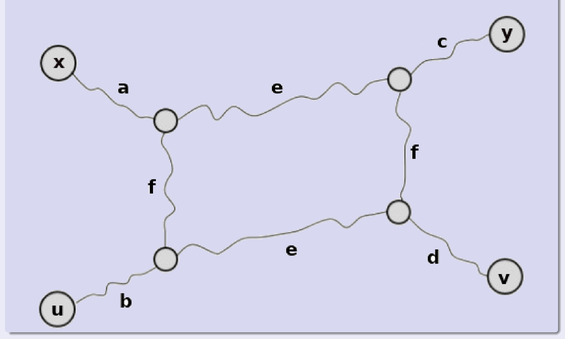
The δ of a graph is the largest δ for all four point combinations.

An example graph that cannot embed into $\mathbb{R}^n$



Four Point Condition Example and Our Calculations





Connections greater than	Number of Nodes	δ	Diameter
200	3	0	1
190	4	0	1
180	6	0.5	2
140	12	0.5	2
100	33	0.5	2
80	54	0.5	2
60	79	1	3
20	287	1	4
15	398	1	5
10	538	1	∞
9	594	1	6
8	638	1	6
7	692	1	7
6	781	1	7
5	915	1	∞
4	1106	1	8
3	1398	1.5	∞
2	1874	1.5	∞
1	2880	1.5	15
0	3618	1.5	17

Conclusions/Future Work

We have discovered that we cannot find an isometric embedding of our Airline Network into $\mathbb{R}^n$. Even using a sub-graph of nodes with degree ≥ 180, we cannot embed this into $\mathbb{R}^n$. Using a similarity embedding may be the most achievable embedding in a useful dimension.

In the future we would like to see if we can relax the requirement for isometric preservation of the distance between nodes would we be able to embed this network into $\mathbb{R}^n$. We would specify that connected nodes have a distance ≤ 1 and disconnected nodes have a distance > 1. We are also exporing what algorithms can be used to embed this network into $\mathbb{H}^n$ since complex networks do not embed isometrically into $\mathbb{R}^n$.

Acknowledgments

We would like to thank our Faculty Mentor, Matt Holzer for leading this research project for us. Without his guidance and direction we would not have been able to discover that the network we were testing with is Hyperbolic and a good candidate for embedding into Hyperbolic space. We greatly look forward to continuing research in this are under his guidance.

We would also like to give thanks to Tyrus Berry for assisting us in the computations for our network and providing insight into Principal Component Analysis and Multidimensional Scaling.

References

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