Wave Fronts in DTDS Population Models

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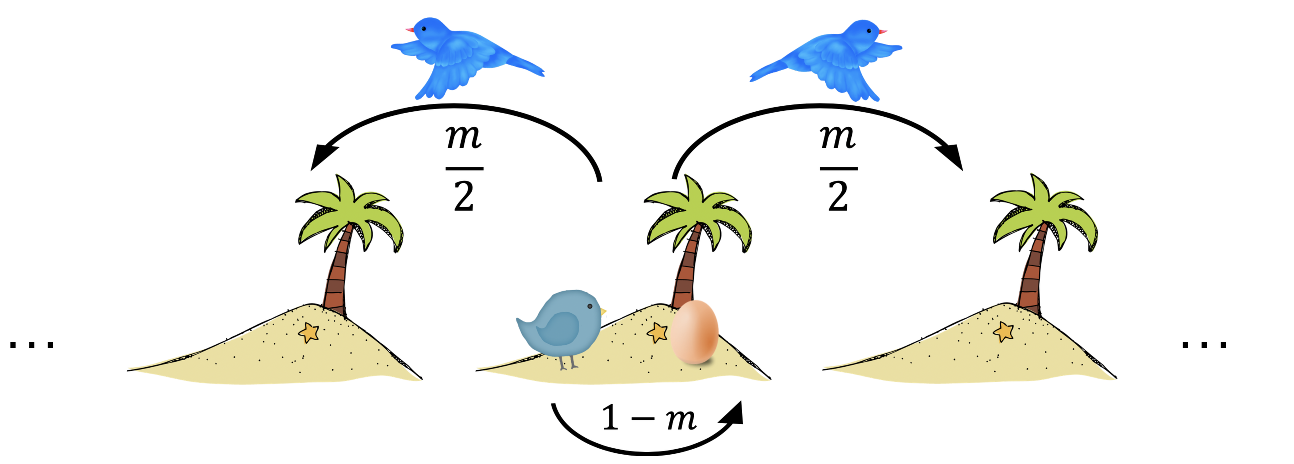
May 8, 2020

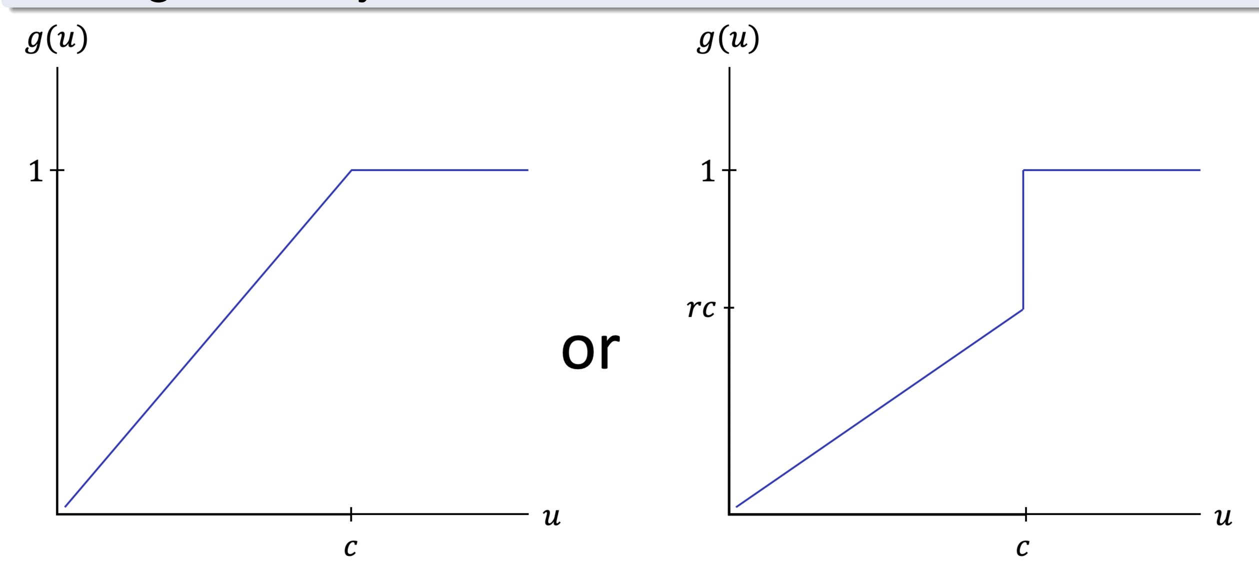
Introduction

- The aim of this research is to study the occurrence of "wave fronts" in population growth models.
- In these models, both time and space can be treated as discrete or continuous.
- The model we focus on is a **discrete-time, discrete-space** (DTDS) model with a linear growth rate.



Description of the Model

- Population sites are nodes on an infinite one-dimensional lattice.
- Each node has a continuous population value $u_{i,t} \in [0,1]$, where i denotes the lattice site, and t denotes the generation number.
- Each generation, the populations **migrate** and then **grow**, according to the function $u_{i,t+1} = g(\frac{m}{2}u_{i-1,t} + (1-m)u_{i,t} + \frac{m}{2}u_{i+1,t})$, where m is the **migration rate**, and g is the **growth function**.
- The migration rate $m \in [0,1]$ is the proportion of $u_{i,t}$ that migrates away from the node.

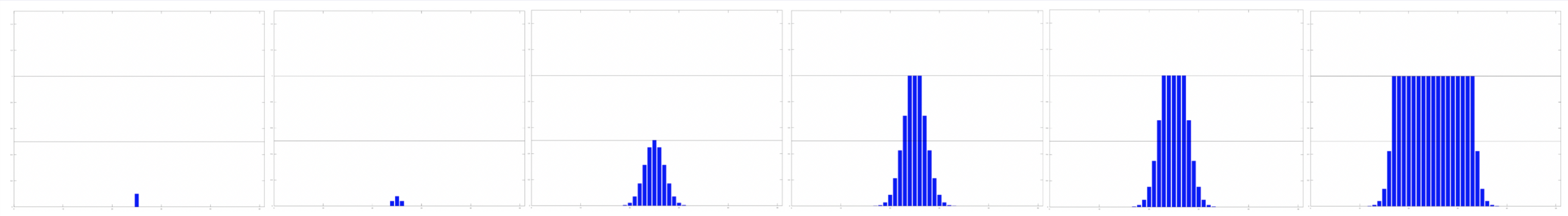


Growth Function

- We focus on a capped linear growth function, given by $g(u) = \begin{cases} ru, & u < c \\ 1, & u \geq c \end{cases}$, and we impose $rc \leq 1$.
- The value r is the **reproduction rate** and, for our purposes, $r > 1$.
- The value $c \in [0,1]$ is the **critical size**, after which a node's population jumps to capacity.

Waves

- As nodes migrate and grow, more and more will reach capacity.
- These nodes form a **wave** that spreads in both directions.
- Without loss of generality, we consider right moving waves.



Wave Fronts

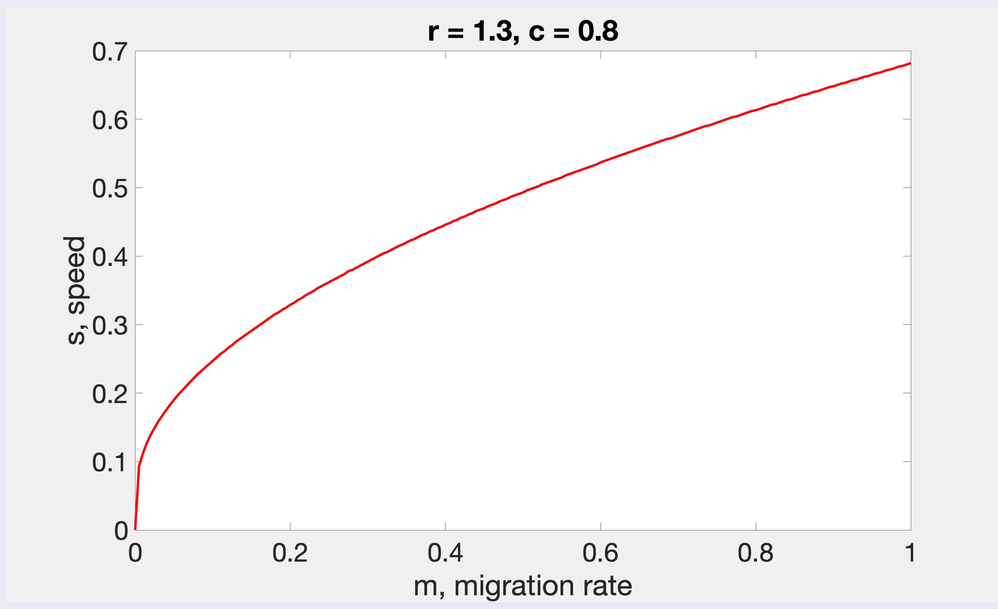
- The **wave front** is the most rightward node with a population of 1. The wave front $d(t)$ is given by $d(t) = \max_{j \in \mathbb{Z}}\{u_{j,t} = 1\}$.
- If we start with an initial population of $u_{0,0} = 1$ and $u_{i,0} = 0$ for $i \neq 0$, then $d(t)$ is **how far the wave has traveled** in t generations.

Wave Speeds

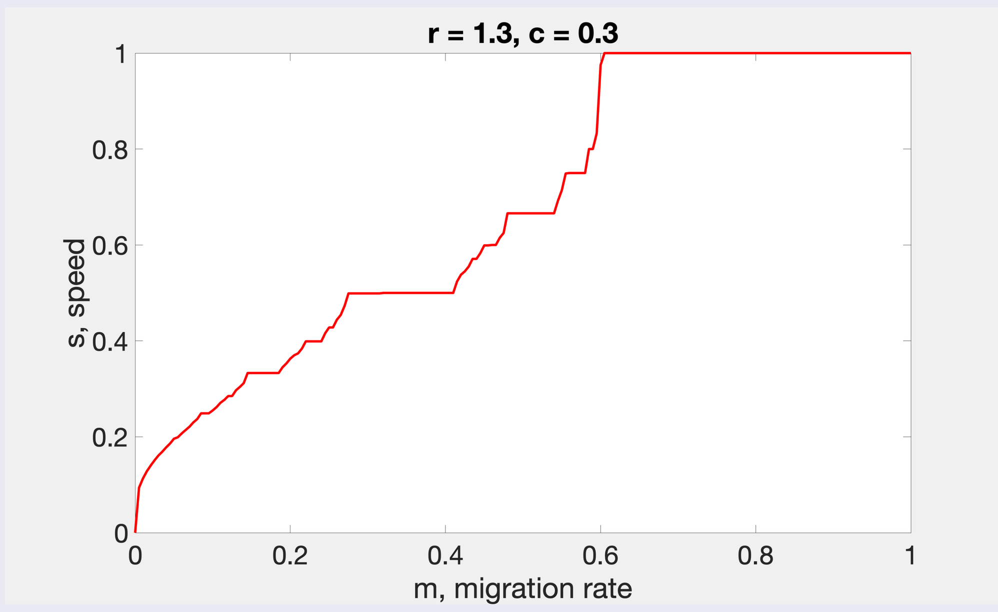
- The **speed** of a wave is defined as $s = \lim_{t \to \infty} \frac{d(t)}{t}$.
- This is essentially distance divided by time.
- We take the limit as t goes to infinity to avoid any initial turbulence.

- Main Idea: **How does wave speed depend on our choice of parameters?**
- (Specifically, the parameters r, m, and c)

Types of Waves



Speed as a function of migration rate. Exhibits **pulled** behavior.

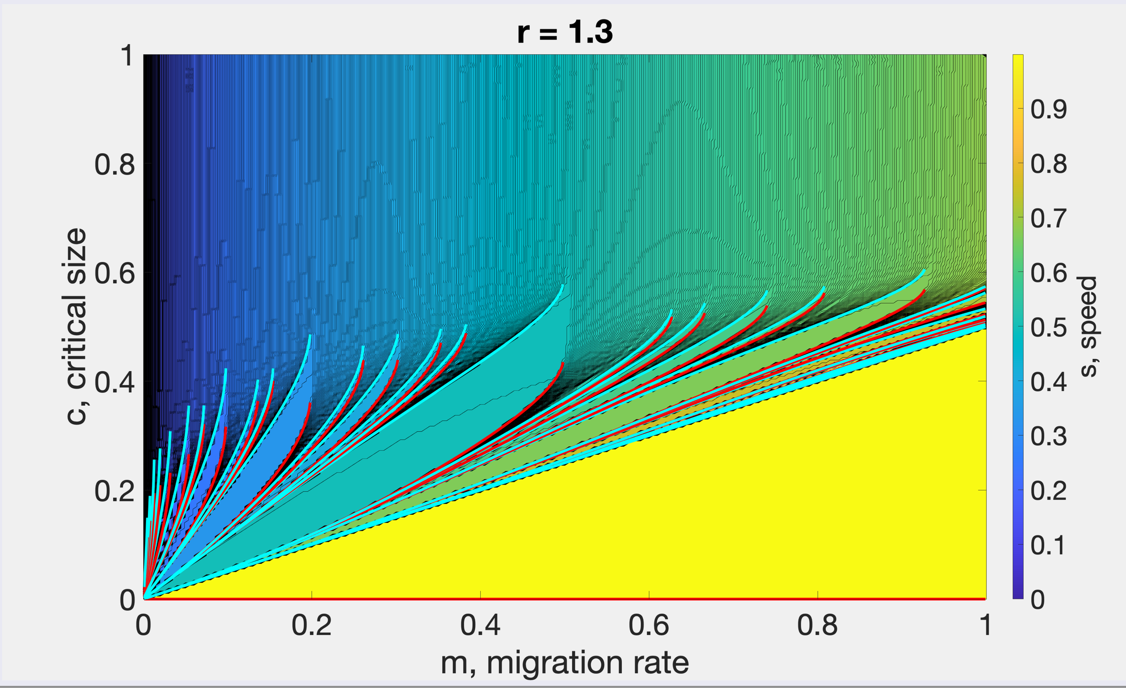


Speed as a function of migration rate. Exhibits **locked** and **pushed** behavior.

- **Pulled Waves**
  - For some waves, **speed varies smoothly** as we vary parameters.
  - One such type of wave are the waves in the form $u_{i,t} = \lambda^t \gamma^i$, with $\lambda > 1$ and $0 < \gamma < 1$.
  - This **uncapped** wave has speed $s = -\frac{\ln(\lambda)}{\ln(\gamma)}$.
  - Pulled waves are ones that move at the speed $s_{lin} = \min_{0<\gamma<1}\left\{-\frac{\ln(\lambda)}{\ln(\gamma)}\right\}$.
- **Locked Waves**
  - For some parameter values, varying the parameters has **no effect on speed**.
  - We call this phenomenon **speed locking**.
  - Each locked region corresponds to a **rational speed**.
  - This project focuses on describing what these waves look like.
- **Pushed Waves**
  - These waves have a speed **greater** than s_{lin}.
  - They correspond to the non-flat regions in between locked regions.
  - We know the least about these waves.

Results

- We have shown that a locked wave with rational speed $s = p/q$ has the form $u_{i,t} = \min\left(1, \Gamma\left(i - \frac{p}{q}t\right)\right)$, where $\Gamma(n) = \sum_{j=1}^{q-p} k_j \gamma_j^n$.
- Where γ_j are the $q-p$ smallest complex solutions of $\left(\frac{rm}{2} + r(1-m)\gamma + \frac{rm}{2}\gamma^2\right)^q = \gamma^{q-p}$.
- And $k_j = \prod_{n \neq j} \frac{\gamma_n^{-1/q} - 1}{\gamma_n^{-1/q} - \gamma_j^{-1/q}}$.
- The region where this wave speed exists is bounded by $\frac{1}{r}\Gamma\left(\frac{1}{q}\right) < c \leq \frac{m}{2} + (1-m)\Gamma\left(\frac{p}{q}\right) + \frac{m}{2}\Gamma\left(1 + \frac{p}{q}\right)$.
- These conditions match what we see in simulations.



Future Work

- What do irrational speed waves look like?
- Can we describe pushed and pulled waves analytically as we have done for locked waves?
- How much of the space do locked waves and push waves occupy?
- As we vary parameters, how does the system transition from exhibiting pulled behavior to exhibiting locked/pushed behavior?

References

[1] C.-H. Wang, S. Matin, A. B. George, and K. S. Korolev, "Pinned, locked, pushed, and pulled traveling waves in structured environments," 2018.
[2] G. Birzu, O. Hallatschek, and K. S. Korolev, "Fluctuations uncover a distinct class of traveling waves," Proceedings of the National Academy of Sciences, vol. 115, no. 16, pp. E3645–E3654, 2018.
[3] W. VANSAARLOOS, "Front propagation into unstable states," Physics Reports, vol. 386, p. 29–222, Nov 2003.
[4] S. R. Gandhi, E. A. Yurtsev, K. S. Korolev, and J. Gore, "Range expansions transition from pulled to pushed waves as growth becomes more cooperative in an experimental microbial population," Proceedings of the National Academy of Sciences, vol. 113, no. 25, pp. 6922–6927, 2016.