Capillary Rise in Porous Materials

Matthew Kearney, Zach Richey, Laura Nicholson

Mason Experimental Geometry Lab

George Mason University

April 30, 2021

Introduction to Capillary Rise

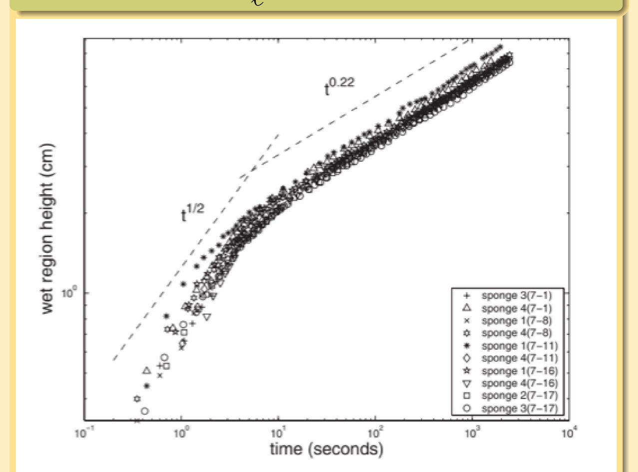
Capillary rise is the tendency of a fluid to rise in a tube or porous medium as a result of that fluid's surface tension. Our goal is to model capillary rise in porous materials.

Dynamics in Porous Materials

- Washburn[5] describes porous media as tuble-bundles, accurate for small time scales.
- Hypothesis: We can more accurately describe fluid rise at longer time scales using a **mixture-theory** model.

Washburn (1921)

$h_\ell \propto t^{1/2}$



Siddique, Anderson, Bondarev[4]

Mixture Theory: Overview

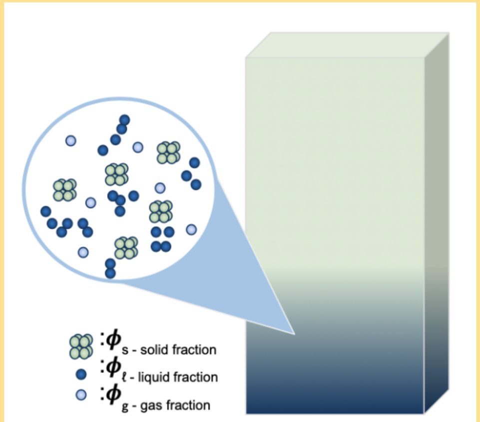


Figure: A partially saturated sponge.

- Represents each point in medium as a small volume containing mixture of phases.

$$\phi_\ell + \phi_g + \phi_s = 1$$

$$S := \frac{\phi_\ell}{1-\phi_s}$$

Classical Results: Brooks & Corey (1964)[1]

- Account for partial saturation and air/fluid pressure differential
- Assume gas/liquid phases are immiscible, flow within continuous porous network

Capillary Pressure

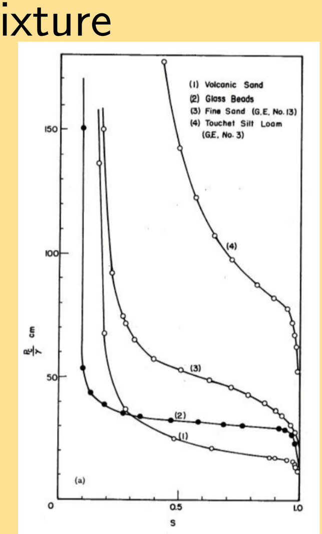
$$p_c(S) = p_c^0\left(\frac{1-S_r}{S-S_r}\right)^{1/\lambda}$$

Relative Permeability

$$k_{r\ell}(S) = \left(\frac{S-S_r}{1-S_r}\right)^{3+2/\lambda}$$

Irrigation & Drainage

- Studied Volcanic Sand, Fine Sand, Glass Beads, Touchet Silt Loam, Sandstone, and Clay/Sandstone/Sand Mixture



Plot of Capillary Pressure Head as function of Saturation[1]

Mixture Theory: Analysis

- Assume only **vertical flow**
- Mass balance:

$$\frac{\partial \phi_\ell}{\partial t} = -\frac{\partial}{\partial z}(w_\ell \phi_\ell)$$

- Momentum balance (Darcy's Law):

$$\frac{\partial p_c}{\partial z} = \frac{K_{s\ell}}{\phi_\ell} w_\ell + \Delta\rho g$$

- Let h_ℓ be the height of the liquid fraction:

$$\frac{dh_\ell}{dt} = w_\ell(z = h_\ell)$$

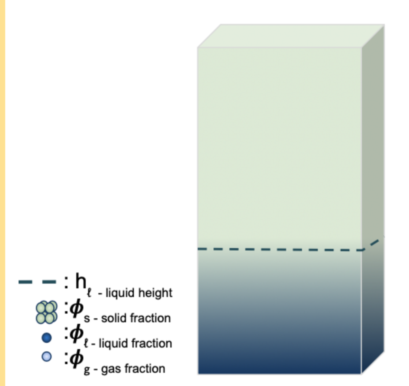


Figure: A partially saturated sponge.

- w_ℓ: liquid velocity
- $\Delta\rho = \rho_\ell - \rho_g$: density difference
- $p_c = p_\ell - p_g$: capillary pressure
- $K_{s\ell}$: friction coefficient

- Boundary Conditions:

$$\phi_\ell(z=0) := \phi_\ell^0$$
$$\phi_\ell(z=1) := \phi_\ell^1$$

- System of Equations:

$$\frac{\partial \phi_\ell}{\partial t} = -\frac{\partial}{\partial z}\left[\frac{\phi_\ell^2}{K_{s\ell}}\left(\frac{\partial p_c}{\partial z} - \Delta\rho g\right)\right]$$

$$\frac{dh_\ell}{dt} = \frac{\phi_\ell}{K_{s\ell}}\left(\frac{\partial p_c}{\partial z} - \Delta\rho g\right)\Bigg|_{z=h_\ell}$$

- Nondimensionalization Parameters:

$$z = \left(\frac{p_c(\phi_\ell^0)}{\Delta\rho g}\right)\bar{z}, \quad t = \frac{\mu}{k_0 p_c(\phi_\ell^0)}\left(\frac{p_c(\phi_\ell^0)}{\Delta\rho g}\right)^2 \bar{t},$$

$$h_\ell(t) = \left(\frac{p_c(\phi_\ell^0)}{\Delta\rho g}\right)\bar{h}_\ell(t), \; p_c(\phi_\ell) = \left(\frac{1}{p_c(\phi_\ell^0)}\right)\bar{p}_c(\phi_\ell)$$

- Dimensionless form:

$$\frac{\partial \phi_\ell}{\partial \bar{t}} = \frac{\partial}{\partial \bar{z}}\left[k_{r\ell}(\phi_\ell)\left(-\frac{\partial \bar{p}_c}{\partial \phi_\ell}\frac{\partial \phi_\ell}{\partial \bar{z}} + 1\right)\right]$$

$$\frac{d\bar{h}_\ell}{d\bar{t}} = -\frac{k_{r\ell}(\phi_\ell)}{\phi_\ell}\left(-\frac{\partial \bar{p}_c}{\partial \phi_\ell}\frac{\partial \phi_\ell}{\partial \bar{z}} + 1\right)\Bigg|_{\bar{z}=\bar{h}_\ell}$$

- Assume flow is **quasi-steady**, i.e. $\frac{\partial \phi_\ell}{\partial \bar{t}} = 0$

$$C(\bar{t}) = k_{r\ell}(\phi_\ell)\left(-\frac{\partial \bar{p}_c}{\partial \phi_\ell}\frac{\partial \phi_\ell}{\partial \bar{z}} + 1\right)$$

$$\frac{d\bar{h}_\ell}{d\bar{t}} = -\frac{C(\bar{t})}{\phi_\ell^1}$$

Quasi-Steady Flow

- Functional forms (from Brooks & Corey):

$$-\frac{\partial \bar{p}_c}{\partial \phi_\ell} = \alpha\gamma(\alpha\phi_\ell + \beta)^{-1-\gamma}$$

$$k_{r\ell}(\phi_\ell) = (\alpha\phi_\ell + \beta)^{3+2\gamma}$$

Where $\alpha = \frac{1-\phi_s}{1-S_r}$, $\beta = \frac{-S_r}{1-S_r}$, and $\gamma = \frac{1}{\lambda}$

- $$C(\bar{t}) = \alpha\gamma(\alpha\phi_\ell + \beta)^{2+\gamma}\frac{\partial \phi_\ell}{\partial \bar{z}} + (\alpha\phi_\ell + \beta)^{3+2\gamma}$$

Integrate using separation of variables

$$\frac{d\bar{h}_\ell}{d\bar{t}} = -\frac{C(\bar{t})}{\phi_\ell^1} \quad (1)$$

$$h_\ell(\bar{t}) = \int_{\phi_\ell^0}^{\phi_\ell^1} \frac{\alpha\gamma(\alpha\phi_\ell + \beta)^{2+\gamma}}{C(\bar{t}) - (\alpha\phi_\ell + \beta)^{3+2\gamma}} d\phi_\ell \quad (2)$$

$$\phi_\ell(\bar{z}=0) := \phi_\ell^0, \quad \phi_\ell(\bar{z}=\bar{h}_\ell) := \phi_\ell^1$$

- Suppose **gravity is negligible**, i.e. $g = 0$:
- Then:

$$\bar{h}_\ell(\bar{t}) = \sqrt{\bar{t}\left(\frac{\gamma(\alpha\phi_\ell^0 + \beta)^{3+\gamma} - \gamma(\alpha\phi_\ell^1 + \beta)^{3+\gamma}}{\phi_\ell^1(3+\gamma)}\right)}$$

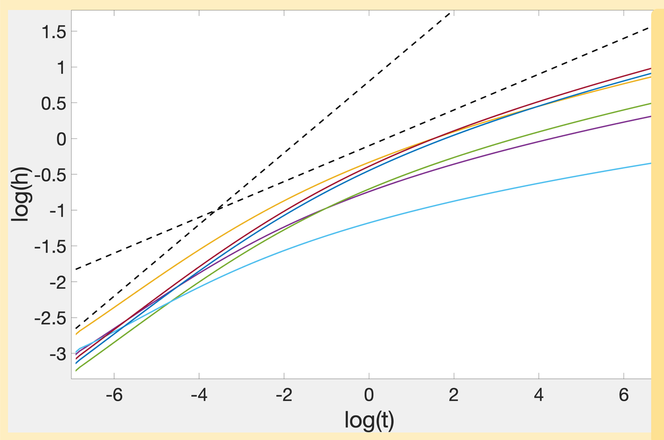
- Equilibrium Height: $\frac{d\bar{h}_\ell}{d\bar{t}} = 0$ occurs when $C(\bar{t}) = 0$.
- This implies:

$$\bar{h}_\ell^{eq} = \frac{1}{(\alpha\phi_\ell^1 + \beta)^\gamma} - \frac{1}{(\alpha\phi_\ell^0 + \beta)^\gamma}$$

Numerical Simulation: Quasi-Steady Flow

- Let $F(\bar{h}_\ell) = C$ be the function given implicitly by (2)

```
t̄ ← t̄_0;  h̄_ℓ(0) ← h̄_ℓ^0;  C(0) ← C_0
While t̄ < t̄_max:
    h̄_ℓ ← h̄_ℓ − (C/φ_1)dt̄  //Euler's method
    C ← F(h̄_ℓ) //Solve with Newton's method
    t̄ ← t̄ + dt̄
```

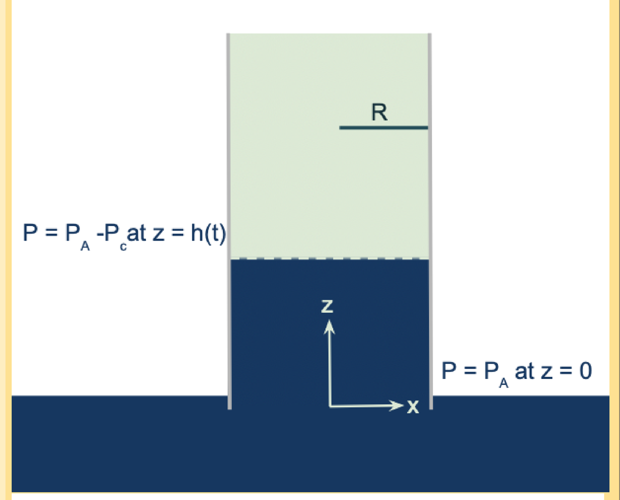


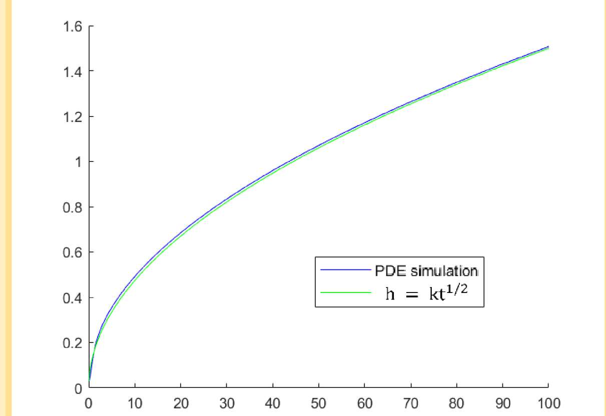
- Consistent with model in zero-gravity case
- Equilibrium height consistent with model
- Similar shape as seen in experiments, but **no clear $t^{1/2}$ or $t^{1/4}$ regimes**

Numerical Simulation: Unsteady Flow

- Examine flow of liquid confined between two parallel plates
- Discretize into a system of ODE's
- Assume that the meniscus is approximately flat, and that the capillary pressure is constant
- Preserves Washburn assumptions but allow fluid velocity to vary over time.
- Assume that

$$\frac{\partial P}{\partial z} = -\frac{P_c}{h(t)}$$





Future Directions

- Further exploration using **numerical simulation**
- Use **asymptotic & perturbation methods** to find an analytic solution
- Attempt to solve without relying upon **quasi-steady assumption**
- Incorporate **material deformation**

Acknowledgments

We would like to thank our project mentor, Dr. Anderson. We would also like to thank Dr. Siddique, who joins us from Penn State; Matthew South, our Graduate Research Assistant; and MEGL for supporting our research.

References

- R. Brooks and A. Corey. Hydraulic properties of porous media. *Hydrology Papers*, 3:11–27, 1964.
- T. Delker, D. Pengra, and P. Wong. Interface pinning and the dynamics of capillary rise in porous media. *Physical Review Letters*, 76(16):2902–2905, 1996.
- M. Lago and M. Araujo. Capillary Rise in Porous Media. *Journal of Colloid and Interface Science*, 234(1):35–43, Feb. 2001.
- J. I. Siddique, D. M. Anderson, and A. Bondarev. Capillary rise of a liquid into a deformable porous material. *Physics of Fluids*, 21(1):013106, 2009.
- E. Washburn. The dynamics of capillary flow. *Physical Review*, 17(3):273–283, 1921.