Finding Line Bundle bases in Equivariant K-theory

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Flags

- A (complete) flag of $\mathbb{C}^n$ is a chain of inclusions of vector subspaces $\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n$, where $\dim V_k = k$.
- The set $\mathrm{Fl}(\mathbb{C}^n)$ of all such flags has the natural structure of both a complex manifold and a complex algebraic variety.
- There is a natural action of an n-dimensional torus T on $\mathrm{Fl}(\mathbb{C}^n)$.

Vector Bundles

- A vector bundle of rank k over $\mathrm{Fl}(\mathbb{C}^n)$ is a morphism $p : E \to \mathrm{Fl}(\mathbb{C}^n)$ such that any fiber of this map has the structure of a $\mathbb{C}$-vector space, and any flag has an open neighborhood X over which there is an isomorphism $\varphi : p^{-1}(X) \to X \times \mathbb{C}^k$ that is linear on each fiber and such that

$$\begin{array}{ccc} p^{-1}(X) & \xrightarrow{\ \varphi\ } & X \times \mathbb{C}^k \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

  commutes, where the downward maps are the obvious ones.
- A T-equivariant vector bundle over $\mathrm{Fl}(\mathbb{C}^n)$ is a vector bundle p with a specified T-action on the total space E such that p is then an equivariant map.

- The set of T-equivariant vector bundles on $\mathrm{Fl}(\mathbb{C}^n)$ is *almost* a ring, under the operations of direct sum and tensor product.
- Two bundles are said to be stably isomorphic whenever they are isomorphic after an addition of a trivial bundle. The corresponding quotient can then be completed to a commutative ring by considering formal differences of bundles. This ring $K_T(\mathrm{Fl}(\mathbb{C}^n))$ is called the T-equivariant K-theory ring of $\mathrm{Fl}(\mathbb{C}^n)$.

Theorem (Kostant-Kumar)

$$K_T(\mathrm{Fl}(\mathbb{C}^n)) \cong \mathbb{Z}[t_1^{\pm}, \cdots, t_n^{\pm}] \otimes_{\mathsf{Sym}} \mathbb{Z}[x_1^{\pm}, \cdots, x_n^{\pm}],$$

$$\text{where } \mathsf{Sym} := \mathbb{Z}[t_1^{\pm}, \cdots, t_n^{\pm}]^{\mathfrak{S}_n}.$$

- This semester we have found a combinatorial basis of this ring as a module over $\mathbb{Z}[t_1^{\pm}, \ldots, t_n^{\pm}]$.

Permutations

- The group $\mathfrak{S}_n$ of permutations on $\{1, 2, 3, \ldots, n\}$ letters is generated by elements that switch consecutive numbers, like $s_2 = [132]$.
- The length of any shortest (reduced) word of a permutation is determined only by the permutation.
- A permutation in $\mathfrak{S}_n$ acts on a polynomial $p \in \mathbb{Z}[x_1, x_2, ..., x_n]$ by permuting the variables $\{x_1, \cdots, x_n\}$ in an obvious way, and we can use this to define divided difference operators ∂_i by

$$\partial_i(p) = \frac{p - s_i \cdot p}{x_i - x_{i+1}}$$

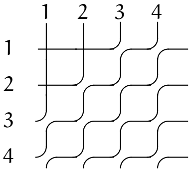
These operators return polynomials, and can be extended to arbitrary permutations in $\mathfrak{S}_n$ by utilizing reduced words.

Schubert Polynomials

- There is a permutation w_0 in $\mathfrak{S}_n$ with the longest length, and we define the Schubert polynomial S_{w_0} to be $x_1^{n-1}x_2^{n-2}...x_{n-1}$.
- We define the Schubert polynomial S_w associated to $w \in \mathfrak{S}_n$ as $\partial_{w^{-1}w_0} S_{w_0}$. These aren't generally monomials.
- The collection of these Schubert polynomials have nice combinatorial properties under the divided difference operators, and form a module basis for $\mathbb{Z}[x_1, \cdots, x_n]$ over the symmetric polynomials.

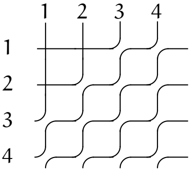
Pipe Dreams

- Each monomial in a Schubert polynomial corresponds to a combinatorial object called a reduced pipe dream.



cont.

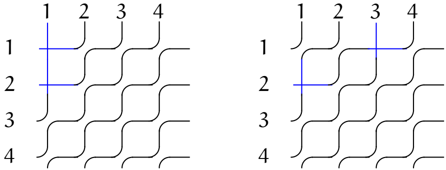
- A pipe dream is a tiling of an $n \times n$ square with 's and 's where any location on or below the antidiagonal is tiled with .
- By following these 'pipes' from the left side to the top, we get a permutation in $\mathfrak{S}_n$



- In this pipe dream, $1 \mapsto 3$, $2 \mapsto 2$, $3 \mapsto 1$ and $4 \mapsto 4$ giving the permutation [3214]

Pipe Dream Data

- To any pipe dream, we can associate a monomial $x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}$ where the exponent of x_i is the number of crosses in the ith row.



- These pipe dreams correspond to different permutations but both give the monomial $x_1 x_2$

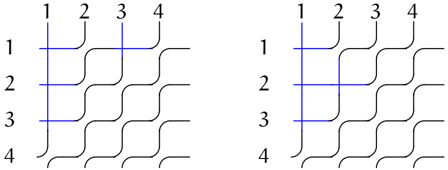
Schubert polynomials and Pipe Dreams

- Pipe dreams and their monomials give us another way to define the Schubert polynomials. We let

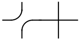
$$S_\pi = \sum_{\pi_p = \pi} \prod_i x_i^{\{\text{\# of crosses in the } i\text{'th row of P}\}}$$

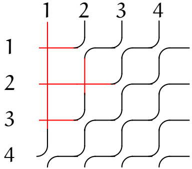
An Example

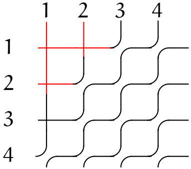
- $S_{[2,4,3,1]} = x_0^2 x_1 x_2 + x_0 x_1^2 x_2$



Flush Left Pipe Dreams

- There is a special pipe dream for a given $\sigma \in \mathfrak{S}_n$ which has the special property that it is flush left, i.e. it doesn't have a block like . These pipe dreams contribute the unique, lexicographically last monomial for each S_π





Construction of Flush Left Monomials

- Similarly, for any monomial $x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}$ of S_π, this monomial corresponds to a flush left pipe dream with e_i crosses in the ith row.
- Let m_σ be the monomial corresponding to σ's unique flush left pipe dream, and let $\sigma_n^> := \text{card}(\{k < n | \sigma(k) > \sigma(n)\})$ be the number of inversions $(\sigma(j), \sigma(n))$. The formula for the monomial corresponding to σ is

$$m_\sigma = \prod x_n^{\sigma(n)-n+\sigma_n^>}.$$

Line Bundle Basis

- The set $M = \{m_\sigma | \sigma \in \mathfrak{S}_n\}$ of flush left pipe dreams, or equivalently the set of lexicographically last monomials in the Schubert polynomials, forms a $\mathcal{Sym}$-basis for the polynomial ring $\mathbb{Z}[x_1^\pm, \ldots, x_n^\pm]$ because they are $\mathbb{Z}$-equivalent to the Schubert basis.

- Translating back to $K_T(\mathrm{Fl}(\mathbb{C}^n))$, we see that the set of elements $1 \otimes m_\sigma$, where $\sigma \in \mathfrak{S}_n$, form a basis for our K-theory ring.

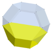
Schubert Varieties

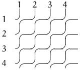
- Inside of our flag variety, we have subvarieties called Schubert varieties that are defined by permutations on the canonical basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$.
- The equivariant K-theory classes generated by these subvarieties are called Schubert classes, and they also form a basis for $K_T(G/B)$. These classes are represented by a polynomial $S_w(t, x)$ of the two lists of variables $t = (t_1, \ldots, t_n)$ and $x = (x_1, \ldots, x_n)$, and are given by

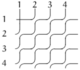
$$S_w(t, x) = \sum_{l(\alpha)+l(\beta)=l(w),\ \alpha=\beta\cdot w} (-1)^{l(\beta)} S_\alpha(t) S_\beta(x).$$

Pipe Dream Decomposition

- Each of these Schubert classes has a decomposition in terms of the monomial basis $\{1 \otimes m_\sigma\}_{\sigma \in \mathfrak{S}_n}$. Our next objective is to give a combinatorial formula for the decomposition of Schubert classes in terms of pipe dreams.



$= t_1 \cdot t_2 \cdot$ 

$- (t_1 + t_2) \cdot$ 

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