Computing Test Ideals of Cohen-Macaulay Modules

Julian Benali, Shrunal Pothagoni

under Dr. Rebecca R.G. at
George Mason University, MEGL

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Preliminaries

All rings will be presumed to be commutative, unital, Noetherian, and local. We will use k to denote a field, R to denote a ring, and M to denote an R-module. We will be working primarily with subrings and quotients of polynomial rings and power series rings.

Definitions

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- A sequence $x_1, ..., x_n$ of elements in R is said to be a regular sequence over M if $(x_1, ..., x_n)M \neq M$ and x_i is not a zero divisor in $M/(x_1, ..., x_{i-1})M$ for all $i \in \{1, ..., n\}$.
  In other words, $(x_1, ..., x_n)M \neq M$ and for all $z \in M$, if $z \notin (x_1, ..., x_i)M$ then $x_{i+1}z \notin (x_1, ..., x_i)M$ as well.

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- We say M is a Cohen-Macaulay (CM) module over R if the length of the longest regular sequence over M is the same as the Krull dimension of R. A finitely generated CM module is called a maximal Cohen-Macaulay (MCM) module.

Definitions

Given any R-module, M, the test ideal of M is

$$\tau_M(R) := \bigcap_{\substack{N, N' \in \mathbf{R\text{-}Mod}, \\ N \subset N'}} (N :_R N_{N'}^{cl_M})$$

where

$$N_{N'}^{cl_M} := \{u \in M : \forall s \in M, s \otimes u \in \operatorname{Im}(S \otimes N \to S \otimes N')\}$$

Definitions

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- Our goal is then to compute the intersection of the test ideals of the MCM modules over R, denoted $\tau_{MCM}(R)$.

Computing Test Ideals

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- Let M be a nonzero free R-module. Then any projection map from M to R is a surjective R-module homomorphism and thus, $\tau_M(R) = R$.
- Every MCM module is a direct sum of indecomposable MCM modules. Furthermore, it follows from the definition that for any CM R-modules, N and L, $\tau_{N \oplus L}(R) \supset \tau_N(R) + \tau_L(R)$.

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From these facts, we conclude that $\tau_{MCM}(R)$ is the intersection of the test ideals of the non-free indecomposable MCM R-modules.

Indecomposable Maximal Cohen-Macaulay Modules

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If a ring has finitely many indecomposable MCM modules up to isomorphism, the ring is said to have finite Cohen-Macaulay type, and if there are countably many, the ring is said to have countable Cohen-Macaulay type.

Known Results

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Thus, calculating the intersection of the test ideals of the CM modules over a ring gives a sense of how close that ring is to being regular.

Sample Code

```
i1 : R=QQ[u,v,w]/ideal(u*w-v^2)
o1 = R
o1 : QuotientRing
i2 : M=module(ideal(u,v))
o2 = image | u v |
o2 : R-module, submodule of R^1
i3 : Hom(M,R)
o3 = image {-1} | v u |
           {-1} | w v |
o3 : R-module, submodule of R^2
```

From this Macaulay2 output, we can read that two homomorphisms which generate $\mathrm{Hom}((u,v), \mathbb{Q}[u,v,w]/(uw-v^2))$ are f_1 and f_2 defined by $f_1(u) = v$, $f_1(v) = w$, $f_2(u) = u$, and $f_2(v) = v$.

Veronese Ring

Let $R = k[\![x^3, x^2y, xy^2, y^3]\!]$. It can be shown that the only indecomposable MCM R-module are $xR + yR$ and $x^2R + xyR + y^2R$ (up to isomorphism). Thus, it is sufficient to compute the test ideal of these modules in order to calculate $\tau_{MCM}(R)$.

The homomorphisms which generate $\operatorname{Hom}(xR + yR, R)$ are f_0, f_1, and f_2 defined by $f_0(p) = x^2p$, $f_1(p) = xyp$, and $f_2(p) = y^2p$ for all $p \in xR + yR$. Then we see $\operatorname{Im}(f_0) = (x^3, x^2y)$, $\operatorname{Im}(f_1) = (x^2y, xy^2)$, and $\operatorname{Im}(f_2) = (xy^2, y^3)$. Thus,

$$\tau_{xR+yR}(R) = (x^3, x^2y) + (x^2y, xy^2) + (xy^2, y^3) = (x^3, x^2y, xy^2, y^3)$$

Veronese Ring

Next, $\mathrm{Hom}(x^2R + xyR + y^2R, R)$ is generated by g_1 and g_2 defined by $g_1(p) = xp$ and $g_2(p) = yp$ for all $p \in x^2R + xyR + y^2R$. Then we see $\mathrm{Im}(g_0) = (x^3, x^2y, xy^2)$ and $\mathrm{Im}(g_1) = (x^2y, xy^2, y^3)$. Thus,

$$\tau_{xR+yR}(R) = (x^3, x^2y, xy^2) + (x^2y, xy^2, y^3) = (x^3, x^2y, xy^2, y^3)$$

So, $\tau_{MCM}(R) = (x^2, x^2y, xy^2, y^3)$, the maximal ideal of R.

Whitney Umbrella (Type D_∞)

Let $R = k[\![x,y,z]\!]/(x^2y+z^2)$, where k is a field of some arbitrary characteristic. Up to isomorphism, the non-free indecomposable MCM R-modules are $cok(zI-\phi)$ where ϕ is one of the following matrices ($j \in \mathbb{Z}^+$):

- $\begin{pmatrix} 0 & -y \\ x^2 & 0 \end{pmatrix}$
- $\begin{pmatrix} 0 & -xy \\ x & 0 \end{pmatrix}$
- $\begin{pmatrix} 0 & 0 & -xy & 0 \\ 0 & 0 & -y^{j+1} & xy \\ x & 0 & 0 & 0 \\ y^j & -x & 0 & 0 \end{pmatrix}$
- $\begin{pmatrix} 0 & 0 & -xy & 0 \\ 0 & 0 & -y^{j} & x \\ x & 0 & 0 & 0 \\ y^j & -xy & 0 & 0 \end{pmatrix}$

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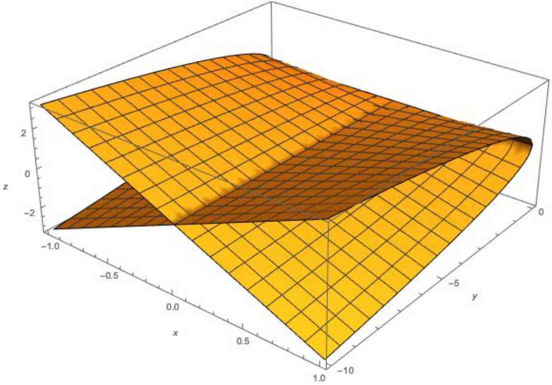
- $\begin{pmatrix} 0 & 0 & -xy & 0 \\ 0 & 0 & -y^j & x \\ x & 0 & 0 & 0 \\ y^j & -xy & 0 & 0 \end{pmatrix}$

We showed that the test ideals corresponding to the matrices are (x^2, y, z), (x, y^j, z), (x, z), and (x, y^j, z) respectively. Thus, $\tau_{MCM}(R) = (x^2, z)$. Note that, unlike in the previous example, this ideal is not m-primary.

Illustration



This illustration shows the Whitney Umbrella which is a surface that is a self intersecting rectangle in R^3. The Whitney Umbrella can be defined by a singular mapping from R^2 to R^3.

Conclusions

Part of the significance of our research was to expand on the result of [Pérez–RG 2019] that $\tau_{MCM}(R)$ is m-primary whenever R has finite Cohen-Macaulay type, but not be m-primary otherwise.The test ideals we have computed give detail on what happens in certain rings which appear frequently as examples.

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- The ring $R[\![x, y]\!]/(x^n + y^2)$ has finite Cohen-Macaulay type for n odd. We determined that all of its tests ideals are m-primary, although some are not m itself.

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- In the Veronese case, we observed that every nontrivial test ideal was the maximal ideal.
- The ring $R[\![x, y]\!]/(x^n + y^2)$ has finite Cohen-Macaulay type for n odd. We determined that all of its tests ideals are m-primary, although some are not m itself.
- For the Whitney Umbrella, which had infinite Cohen-Macaulay type, the intersection of the test ideals was not m-primary.

References

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