The Riemann-Hilbert Correspondence in Categorical Language

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Abstract

The Riemann-Hilbert correspondence for a smooth manifold is the statement that local systems of finite dimensional vector spaces are in bijection with vector bundles with flat connection over M, and moreover this extends to an equivalence of categories. We give a categorical proof of this fact.

Introduction

A common question in topology is how to characterize certain structures that lie "above" a given topological space, say M, or how we can interpret algebraic operations of the homotopy groups on other structures. In the case of an action of the fundamental group on a set S, along with a consistent choice of map $F_\gamma : S \to S$ for each path $\gamma : y \to z$ up to homotopy. As it happens, this is equivalent to constructing a covering space over M!

If we look at a more natural object called the fundamental groupoid $\Pi_1(M)$, we can state this as an equivalence of categories $\mathrm{Fun}(\Pi_1(M), \mathrm{Set}) \simeq Cov/M$

This equivalence is due to the unique path lifting property of covering spaces. If we look at it from the algebraic side, what happens if we change the target category?

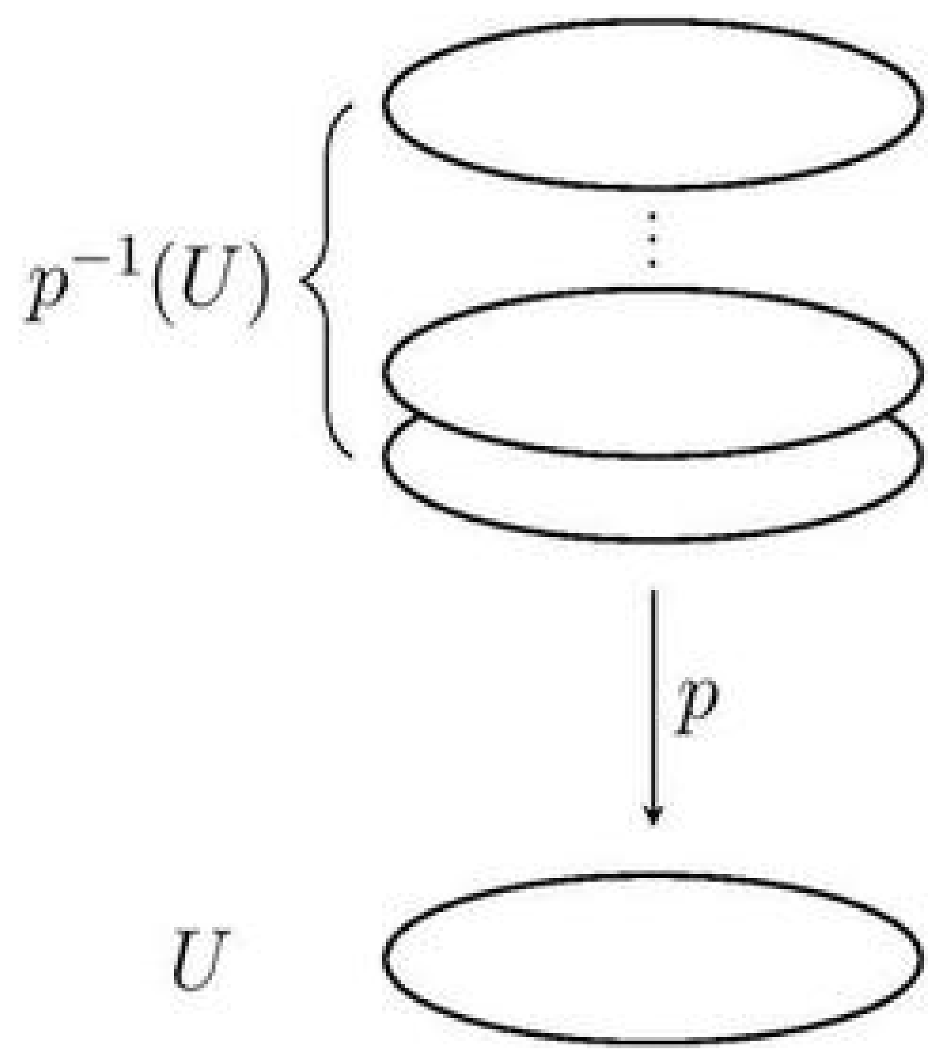

Figure 1: The idea of a covering space

The Categories in Question

$\Pi_1(M)$ is the fundamental groupoid of M, with the set of objects just the set M itself, and with homotopy classes of paths as morphisms.

$\mathrm{Vect}^{\mathrm{f.d.}}_{\mathbb{R}}$ is the category of finite dimensional vector spaces and linear maps between them.

Given two categories $\mathcal{C}$ and $\mathcal{D}$, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the category of functors between them and natural transformations between such functors.

$\mathrm{Bun}_\flat(\mathrm{Vect}^{f.d.}_{\mathbb{R}}, M)$ is the category of vector bundles with homotopy invariant parallel transport, and vector bundle homomorphisms which are "equivariant" with respect to the transport in the following sense: If an element of the bundle is first mapped onto the second and then transported, this is the same as if the element were first transported and then mapped onto the second bundle.

Construction of the Universal Cover

Now, take M a connected smooth manifold, and take $(U_a \hookrightarrow M)_{a\in A}$ a good cover (by which we mean that all finite intersections of the opens are contractible).

For a functor $F : \Pi_1(M) \to \mathrm{Set}$, where $\Pi_1(M)$ is the fundamental groupoid of M, a cover $(U_a \hookrightarrow M)_{a\in A}$ with each inclusion $\pi_1(U_a, x) \to \pi_1(M, x)$ trivial for all $x \in U_a$, and a choice of $x_a \in U_a$ for each open in the cover, We can form the space $\coprod_{a\in A} F(x_a) \times U_a$ using $F(x_a)$ as a discrete space, and form the quotient space $\int_M F$ using the equivalence relation $(u \in F(x_a), y) \sim (v \in F(x_b), z)$ if $y = z$ and there exists a path homotopy class $[\gamma] : x_a \to x_b$ such that $F[\gamma](u) = v$. This carries a natural map p back to M by taking $p([u, x \in U_a]) = x$. This is a covering map, as by construction $p^{-1}(U_a) \cong U_a \times F(x_\alpha)$. We call this space $\int_M F$. Take the functor $\mathrm{Hom}(x, -) : \Pi_1(M) \to \mathrm{Set}$. The action

$$\mu : \pi_1(M, x) \times \int_M \mathrm{Hom}(x, -) \to \int_M \mathrm{Hom}(x, -)$$

is given by

$$([\alpha] : x \to x, [\gamma] : x \to y) \mapsto [\gamma * \alpha] : x \to y.$$

This makes the universal cover a principal bundle.

Going From Local Systems to Flat Bundles

Now, given a functor $F : \Pi_1(M) \to \mathrm{Vect}_{\mathbb{R}}$, we can take the vector bundle $\phi : P_F \to \int_M \mathrm{Hom}(x, -)$ with specified trivialization (which we will give explicitly over a fiber $\phi^{-1}([\gamma] : x \to y) = F(y)$)

$$\rho_y : \phi^{-1}([\gamma]) \to F(x) \ , \vec{v} \mapsto F([\gamma^{-1}])(\vec{v}) \in F(x).$$

$\pi_1(M, x)$ acts on the left of P_F by

$$([\sigma], ([\gamma] : x \to y, \vec{v})) \mapsto ([\gamma * \sigma^{-1}], F[\gamma * \sigma * \gamma^{-1}](\vec{v})).$$

we can take the quotient of P_F by the group action and endow it with a map $\eta : \Omega F \to M$ taking the equivalence class $[[\gamma] : x \to \gamma(1), v]$ to $\gamma(1)$. This gives ΩF a smooth vector bundle structure over M. Moreover, we can then endow ΩF with a homotopy invariant parallel transport. If we have a curve $\gamma : y \to z$, we can just take its image through our functor.

From Flat Bundles to Functors

Going the other way, we can look at vector bundles with flat connection and try to extract algebraic data. Given a bundle with a flat connection, we can construct a parallel transport that is homotopy invariant. Given a vector bundle E with flat connection Π^E (i.e. a vector bundle with homotopy invariant parallel transport), we construct the functor $\Lambda E : \Pi_1(M) \to \mathrm{Vect}^{f.d.}_{\mathbb{R}}$ as follows:

$$\Lambda E(y) := E_y, \Lambda(\gamma : yz) = \Pi^E_\gamma : E_y \to E_z.$$

This is a functor because of the preservation of composition of the parallel transport $(\Pi^E_{\gamma\cdot\alpha} = \Pi^E_\gamma \circ \Pi^E_\alpha)$.

Functoriality

We can not only assign vector bundles to functors, but assign vector bundle homomorphisms to natural transformations $\alpha : F \Rightarrow H$ by taking the map induced by $P_F \Rightarrow P_H$ which in the trivial bundle is just the identity paired with the linear map $\alpha_x : Fx \to Hx$. This map $\Omega\alpha : \Omega F \to \Omega H$ respects the parallel transport. Moreover, this assignment preserves identities and compositions of natural transformations. This makes $\Omega : Fun \to \mathrm{Bun}_\flat(\mathrm{Vect}^{f.d.}_{\mathbb{R}}, M)$ a functor!

In addition, we can assign natural transformations to vector bundles respecting homotopy invariant parallel transports. This is due to the fact that, as mentioned earlier, these maps are equivariant in the same sense as a natural transformation. This in turn makes the assignment Λ a functor.

Equivalence

To ask that these functors

$$\Omega : \mathrm{Fun}(\Pi_1(M), \mathrm{Vect}^{f.d.}_{\mathbb{R}}) \leftrightarrow \mathrm{Bun}_\flat(\mathrm{Vect}^{f.d.}_{\mathbb{R}}, M) : \Lambda$$

are inverse to each other is too stringent a requirement for the categories being the same, so we ask if these functors are isomorphic to the identities in their respective functor categories.

For each functor $F : \Pi_1(M) \to \mathrm{Vect}^{f.d.}_{\mathbb{R}}$, we define a natural transformation $F \overset{\omega_F}{\Rightarrow} \Lambda\Omega F$ as follows:

$$\omega_{Fy}(\vec{v} \in Fy) = [\gamma, \vec{v}].$$

The assignment $F \to \omega_F$ itself composes a natural transformation. Given a natural transformation $\alpha : F \Rightarrow H$, $\omega_H \circ \alpha = \Lambda\Omega\alpha \circ \omega_F$. Again, note that each ω_F is an isomorphism, so this assembles into a natural isomorphism $\omega : \mathrm{id}_{\mathrm{Fun}(\Pi_1(M), \mathrm{Vect}^{f.d.}_{\mathbb{R}})} \Rightarrow \Lambda\Omega$.

As the vector bundle $\Omega\Lambda E$ is constructed as a quotient (colimit) for any E, there is a canonical cocone which induces a map $\lambda_E : \Omega\Lambda E \to E$ which is natural as well. This also assembles into a natural isomorphism $\lambda : \Omega\Lambda \Rightarrow \mathrm{id}_{\mathrm{Bun}_\flat(\mathrm{Vect}^{f.d.}_{\mathbb{R}}, M)}$, and so the functors Ω, Λ are equivalences

References

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