Test Ideals of Cohen-Macaulay Modules

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May 3, 2019

Abstract

In our research, we compute the test ideals of finitely-generated CM (MCM) modules of various rings. The rings we study include subrings and quotients of polynomial rings and power series rings. The test ideals of such rings yield information about the rings' singularities and geometric properties: vaguely, the larger the test ideal, the less singular the ring, and vice versa. If the ring is nonsingular, the test ideal coming from any MCM module is the whole ring.

Introduction

All rings are assumed to be commutative, unital, Noetherian, and local. We will use R to denote a ring, m to denote its maximal ideal, and M to denote an R-module.

- The Krull dimension of R is the length of a longest proper chain of prime ideals $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n$.
- A sequence $x_1, \dots, x_n$ of elements in R is said to be a regular sequence over M if $(x_1, \dots, x_n)M \neq M$ and x_i is not a zero divisor in $M/(x_1, \dots, x_{i-1})M$ for all $i \in \{1, \dots, n\}$.
- We say M is a Cohen-Macaulay (CM) module over R if the length of a longest regular sequence over M is the same as the Krull dimension of R. A finitely generated CM module is called a maximal Cohen-Macaulay (MCM) module.

Test Ideal

If R is a complete local domain, the test ideal of M is

$$\tau_M(R) = \sum_{f \in \mathrm{Hom}_R(M,R)} f(M)$$

That is, we find the images of the R-module homomorphisms from M to R and take their sum [Pérez–RG 2019]. Our goal is then to compute the intersection of the test ideals of the MCM modules, which we denote $\tau_{MCM}(R)$.

Computing Test Ideals

In practice, we do not need to compute the test ideal of every MCM R-module in order to compute $\tau_{MCM}(R)$.

- Let M be a nonzero free R-module. Then any projection map from M to R is a surjective R-module homomorphism and thus, $\tau_M(R) = R$.
- Every MCM module is a direct sum of indecomposable MCM modules. Furthermore, it follows from the definition that for any CM R-modules, N and L, $\tau_{N \oplus L}(R) \supset \tau_N(R) + \tau_L(R)$.

From these facts, we conclude that $\tau_{MCM}(R)$ is the intersection of the test ideals of the non-free indecomposable MCM R-modules.

Indecomposable Maximal Cohen-Macaulay Modules

Finding all the indecomposable MCM modules of a ring is typically a difficult task. So, we took examples for which all the indecomposable MCM modules were known in order to compute $\tau_{MCM}(R)$. If a ring has finitely many indecomposable MCM modules up to isomorphism, the ring is said to have finite Cohen-Macaulay type, and if there are countably many, the ring is said to have countable Cohen-Macaulay type.

Examples

Veronese Ring

Let $R = k[\![x^d, x^{d-1}y, \dots, xy^{d-1}, y^d]\!]$ where k is some field. Up to isomorphism, the non-free indecomposable MCM R-modules are $xR + yR, x^2R + xyR + y^2R, \dots, x^{d-1}R + x^{d-2}yR + \cdots + xy^{d-2}R + y^{d-1}R$. This means R has finite Cohen-Macaulay type.

We showed that if M is one of the above modules, then $\tau_M(R) = m$ where m is the maximal ideal. Thus, $\tau_{MCM}(R) = m$.

Curve Singularity of Type A_{n-1}

Let $R = k[\![x, y]\!]/(x^n + y^2)$ where k is an algebraically closed field and n is some odd positive integer. Up to isomorphism, the non-free indecomposable MCM R-modules are $(x, y), (x^2, y), \dots, (x^{\frac{n-1}{2}}, y)$ (Yoshino p. 39). This means R has finite Cohen-Macaulay type.

We showed that for each $d \in \{1, 2, \dots, \frac{n-1}{2}\}$, $\tau_{(x^d, y)}(R) = (x^d, y)$. Thus, $\tau_{MCM}(R) = (x^{\frac{n-1}{2}}, y)$.

Whitney Umbrella (Type D_∞)

Let $R = k[\![x, y, z]\!]/(x^2y + z^2)$ where k is a field of some arbitrary characteristic. Up to isomorphism, the non-free indecomposable MCM R-modules are $cok(zI - \phi)$ where ϕ is one of the following matrices over $k[\![x, y]\!]$ $(j \in \mathbb{Z}^+)$:

- $\begin{pmatrix} 0 & -y \\ x^2 & 0 \end{pmatrix}$
- $\begin{pmatrix} 0 & -xy \\ x & 0 \end{pmatrix}$
- $\begin{pmatrix} 0 & 0 & -xy & 0 \\ 0 & 0 & -y^{j+1} & xy \\ x & 0 & 0 & 0 \\ y^j & -x & 0 & 0 \end{pmatrix}$
- $\begin{pmatrix} 0 & 0 & -xy & 0 \\ 0 & 0 & -y^j & x \\ x & 0 & 0 & 0 \\ y^j & -xy & 0 & 0 \end{pmatrix}$

(Leuschke and Wiegand p. 256) This means R has countably infinite Cohen-Macaulay type.

We showed that the test ideals corresponding to the matrices are (x^2, y, z), (x, y^j, z), (x, z), and (x, y^j, z) respectively. Thus, $\tau_{MCM}(R) = (x^2, z)$. Note that, unlike in the other examples, this ideal is not m-primary.

Illustrations

Figure 1 represents the example of a simple curve singularity of type A_{n-1}. The illustration shows the case $n = 3$ given by the equation $y^2 = -x^3$. Figure 2 shows the Whitney Umbrella which is a surface that is a self intersecting rectangle in R^3. The Whitney Umbrella can be defined by a singular mapping from R^2 to R^3.

Figure 1

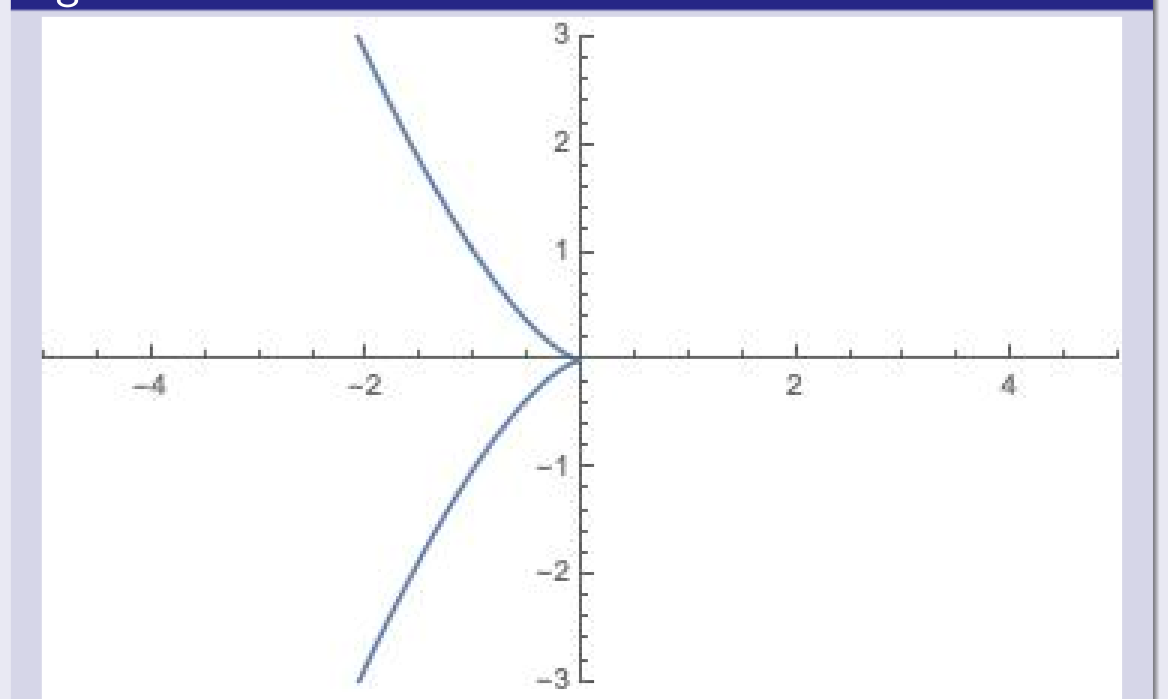
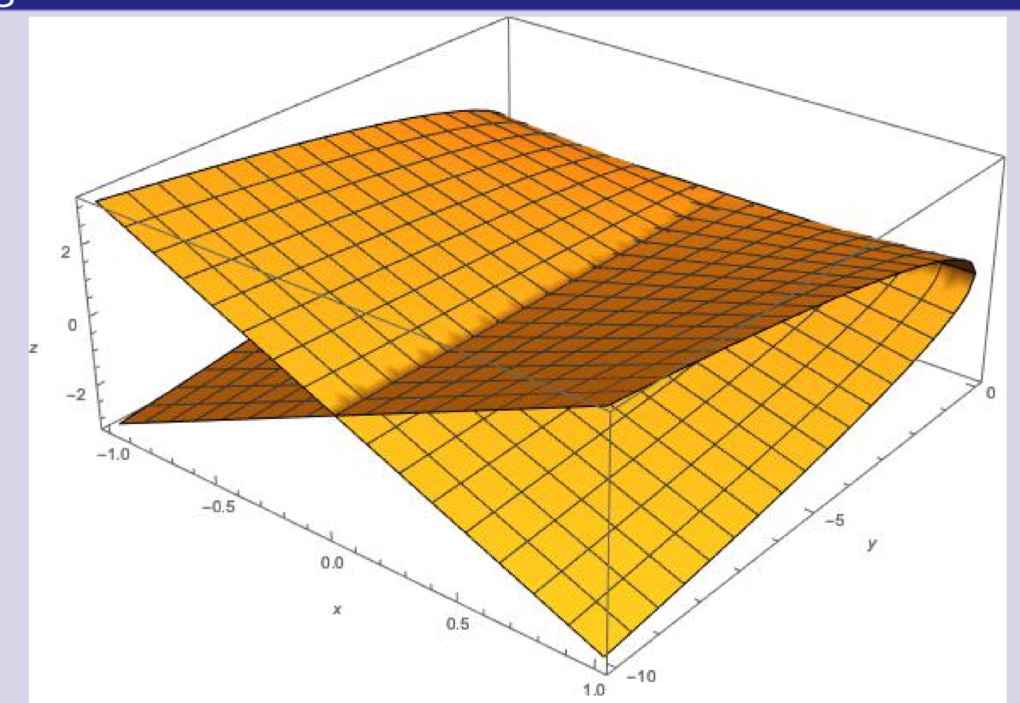


Figure 2



Macaulay2

To find the intersections of our test ideals we used a computer algebra system named Macaulay2. This is software that is specifically devoted to assisting research in algebraic geometry and commutative algebra. Specifically, we used this software to compute the homomorphisms from our indecomposable MCM modules to the ring. Then by hand, we would take the sum of the images of the homomorphisms to obtain the test ideals and then intersect the test ideals.

Macaulay2 Sample Code

```
i1 : R=QQ[u,v,w]/ideal(u*w-v^2)
o1 = R
o1 : QuotientRing
i2 : M=module(ideal(u,v))
o2 = image | u v |
o2 : R-module, submodule of R^1
i3 : Hom(M,R)
o3 = image {-1} | v u |
           {-1} | w v |
o3 : R-module, submodule of R^2
```

From this Macaulay2 output, we can read that two homomorphisms which generate $\mathrm{Hom}((u, v), \mathbb{Q}[u, v, w]/(uw - v^2))$ are f_1 and f_2 defined by $f_1(u) = v$, $f_1(v) = w$, $f_2(u) = u$, and $f_2(v) = v$.

Conclusions

Part of the significance of our research was to expand on the result of [Pérez–RG 2019] that $\tau_{MCM}(R)$ is m-primary whenever R has finite Cohen-Macaulay type, but may not be m-primary otherwise. The test ideals we have computed give detail on what happens in certain rings which appear frequently as examples. In the Veronese case, we observed that every nontrivial test ideal was the maximal ideal. In the A_n singularity, we saw that the test ideals were all m-primary, but not necessarily m itself. However, with the Whitney Umbrella, which had infinite Cohen-Macaulay type, the intersection of the test ideals was not m-primary.

Acknowledgements

We would like to thank the Mason Experimental Geometry Laboratory for supporting us and providing us with the opportunity to conduct our research. We would also like to thank Dr. Rebecca R.G. for her guidance throughout our research.