Change of Basis in the Equivariant K-theory of Flag Manifolds

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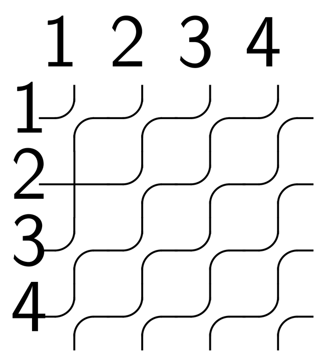
Abstract

This semester, our team investigated the module structure of the equivariant K-theory of a flag manifold $\mathrm{Fl}(\mathbb{C}^n)$, with a natural torus T action, over symmetric polynomials, which we will denote by $K_T(\mathrm{Fl}(\mathbb{C}^n))$. In particular, we sought to find a basis consisting of monomials representing equivariant line bundles over some $Fl(\mathbb{C}^n)$.

We did this using combinatorial tools called pipe dreams, and special pipe dreams called flush left pipe dreams in particular. We verified that this set was a basis using the structure of symmetric polynomials after localization.

Combinatorial Tools

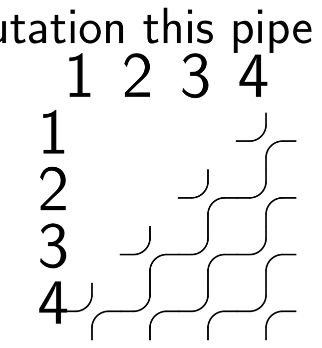
A k-pipe dream is a lot like a square k×k matrix with entries being one of two different symbols, and , with the condition that if a_{ij} is an entry such that $i+j>k$ then $a_{ij} =$.



- A reduced pipe dream will be a pipe dream such that any two "pipes," the paths emanating from the left of each row, cross at most once. We only dealt with reduced pipe dreams, so henceforth pipe dream will mean a reduced pipe dream.
- For any pipe dream, one can obtain a permutation on the set $\{1, 2, \cdots, k\}$ by following the "pipes" from left to above.
- One can also associate to each pipe dream P a certain monomial $m_P := x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ where e_i is the number of crosses in row i.
- A pipe dream is said to be flush left if it doesn't contain a block like , i.e. if there isn't an elbow left of a cross. There are $k!$ of these pipe dreams, each one corresponding to enumeration of crosses in each row.

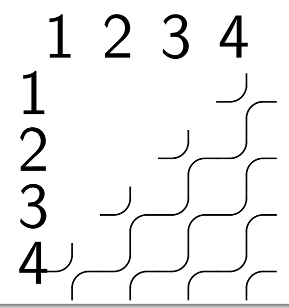
CHALLENGES

Can you tell what permutation this pipe dream corresponds to?



Can you come up with a pipe dream for this permutation?

$(1 \rightarrow \quad , 2 \rightarrow \quad , 3 \rightarrow \quad , 4 \rightarrow \quad)$



Polynomial Bases

We worked with a basis of $K_T(\mathrm{Fl}(\mathbb{C}^n))$ represented by polynomials called Schubert polynomials. These polynomials correspond to permutations in S_n, where

$$\mathfrak{S}_\sigma := \sum_{\text{P a pipe dream for } \sigma} m_P.$$

Our work this semester was to refine this to a monomial basis, using flush left pipe dreams. Each flush left pipe dream contributes to some $\mathfrak{S}_\sigma$, and we will define M_σ to be the monomial corresponding to this pipe dream. This monomial is also the first monomial of $\mathfrak{S}_\sigma$ in the reverse lexicographic order.

A Concrete Monomial Basis

The definition of M_σ is well defined, as any two flush left pipe dreams (having different monomials) have different corresponding permutations. The collection of such M_σ in fact also form a basis for our equivariant K-theory

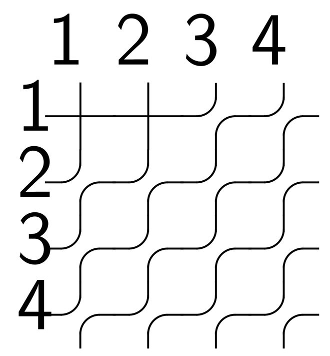
In addition, we have also found a formula to compute any M_σ. We have

$$M_\sigma = \prod x_n^{\sigma(n)-n+\sigma_n^>}$$

where

$$\sigma_n^> := \mathrm{card}(\{k < n \mid \sigma(k) > \sigma(n)\})$$

. Take the permutation [3124]. Using our formula, we can see that $M_{[3124]} = x_1^2$, and the corresponding pipe dream is

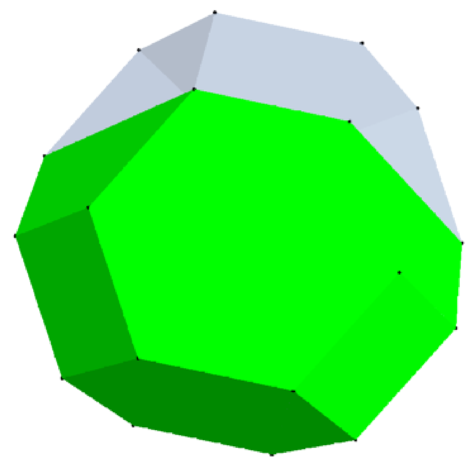


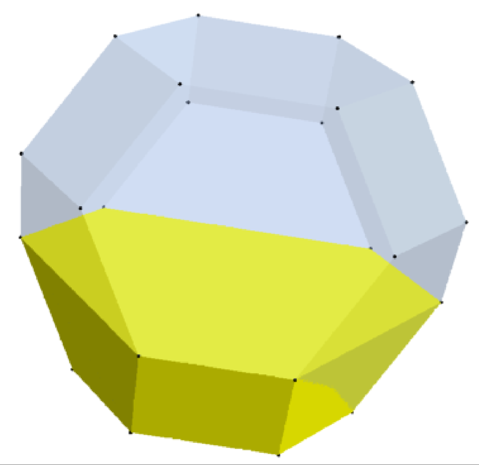
Geometrical Interpretation

Theorem (Kostant-Kumar) : $K_T(\mathrm{Fl}(\mathbb{C}^n)) \cong R \otimes_A R$ where $R = \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \cdots x_n^{\pm 1}]$ and A is the subring of symmetric polynomials. Our monomials $\{M_\sigma\}$ become elements $1 \otimes M_\sigma \in K_T(\mathrm{Fl}(\mathbb{C}^n))$. Geometrically, these correspond to a line bundle over $\mathrm{Fl}(\mathbb{C}^n)$ with an action of T that respects the action of T "downstairs," on the base space $\mathrm{Fl}(\mathbb{C}^n)$.

That these monomials form a basis for the ring means that every formal difference of equivariant vector bundles is, up to a suitable notion of equivalence, the same as a sum of equivariant line bundles.

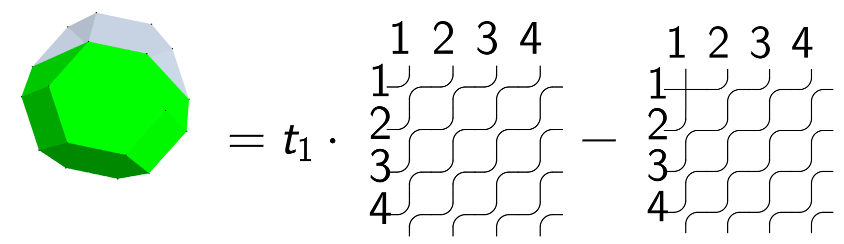
Along with our bases of Schubert polynomials and monomials, there are other more immediately geometrically motivated bases corresponding to subspaces of $\mathrm{Fl}(\mathbb{C}^n)$ called Schubert varieties. Below are representations of some Schubert varieties inside of the flag variety $\mathrm{Fl}(\mathbb{C}^4)$.



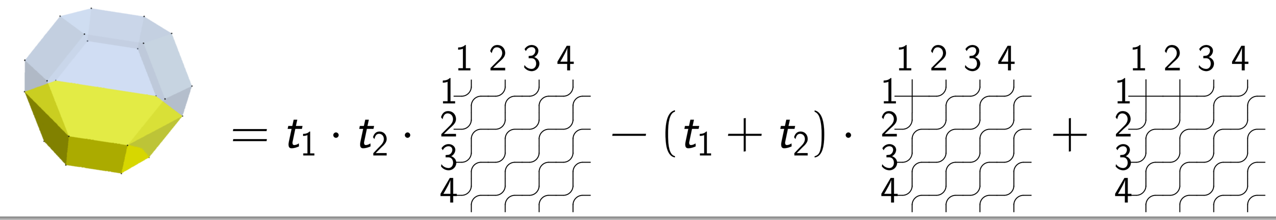


Translating Between Varieties and Monomials

We wanted to translate between the equivariant K-theory classes of the Schubert varieties and the basis of line bundles that we had obtained. These Schubert classes have a representation through what are called double Schubert polynomials, and thus can be easily compared to our monomials and pipe dreams. For example, the Schubert variety corresponding to [2134] has the expression $t_1 - x_1$. Properly interpreting this in the K-theory ring as $t_1 \otimes 1 - 1 \otimes x_1$, we can decompose the K-theory class as



We can do likewise with



Acknowledgements

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References

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Kostant, Bertram. Kumar, Shrawan. T-Equivariant K-theory of Generalized Flag Varieties. Proceedings of the National Academy of Sciences of the United States of America. 1987 Jul; 84(13): 43514354.