On Two Bases of Equivariant K-theory of Flag Manifolds

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Introduction

Definition

A **complete flag** in $\mathbb{C}^n$ is a chain of vector subspaces

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{C}^n$$

where the (complex) dimension of V_i is i for all $0 \le i \le n$.

For the purposes of this talk we'll just refer to a complete flag as a flag.

Definition

The (complete) **flag manifold** $Fl(n;\mathbb{C})$ is the space of all complete flags in $\mathbb{C}^n$.

Some Definitions

It is convenient to represent a flag as an invertible $n \times n$ matrix A where the subspace V_i is the span of the first i columns of X. For example, in $\mathbb{C}^3$ the matrix

$$g = \begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{2,1} & g_{2,2} & g_{2,3} \\ g_{3,1} & g_{3,2} & g_{3,3} \end{pmatrix}$$

represents the flag

$$\{0\} \subset \operatorname{span}\left\{\begin{pmatrix} g_{1,1} \\ g_{2,1} \\ g_{3,1}\end{pmatrix}\right\} \subset \operatorname{span}\left\{\begin{pmatrix} g_{1,1} \\ g_{2,1} \\ g_{3,1}\end{pmatrix}, \begin{pmatrix} g_{1,2} \\ g_{2,2} \\ g_{3,2}\end{pmatrix}\right\} \subset \operatorname{span}\left\{\begin{pmatrix} g_{1,1} \\ g_{2,1} \\ g_{3,1}\end{pmatrix}, \begin{pmatrix} g_{1,2} \\ g_{2,2} \\ g_{3,2}\end{pmatrix}, \begin{pmatrix} g_{1,3} \\ g_{2,3} \\ g_{3,3}\end{pmatrix}\right\} = \mathbb{C}^3$$

Different matrices

- Lots of different matrices represent the same flag.

Definition

Let $G = GL(n;\mathbb{C})$ be the **general linear group** of invertible $n \times n$ complex valued matrices (under matrix multiplication).

Definition

Let B denote the subgroup of upper triangular matrices.

Remark. For any $g \in G$ and for any $b \in B$, b and $g \cdot b$ have the same column spans.

A Group Action on Flag Manifolds

Definition

Let $T^n \simeq S^1 \times S^1 \times \cdots \times S^1$ be the **torus** of dimension n.

- We will realize T^{n-1} as a diagonal $n \times n$ matrix where $a_{j,j} = e^{i\theta_j}$ for $1 \le j \le n-1$, and $a_{n,n} = e^{i(\theta_1 + \cdots + \theta_{n-1})}$.
- One purpose as expressing $Fl(n;\mathbb{C})$ as G/B is realizing a left T^{n-1} action on $Fl(n;\mathbb{C})$.
- T acts on G/B by $t(gB) = (t \cdot g)B$.

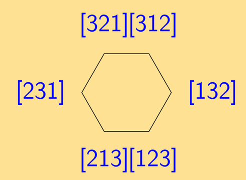
Fixed Points

- The fixed points of this action are of particular interest.
- Which flags are fixed for all $t \in T$?

Remark. The set of T-fixed points of $Fl(n;\mathbb{C})$, denoted $Fl(n;\mathbb{C})^T$, is indexed by S_n, the symmetric group on $1, \ldots, n$. The fixed flags are realized as permutation matrices.

T^2 acts on $Fl(3;\mathbb{C})$; the fixed point set is indexed by

$$S_3 = \{[123], [213], [132], [231], [312], [321]\}.$$



Injection to the Fixed Points

Theorem

(Steinberg, Kostant-Kumar)

$$K_T(Fl(n;\mathbb{C})) \simeq R(T) \otimes_{R(T)^{S_n}} R(T)$$

where $R(T)$ is the ring of representations of T, and:

$$R(T^n) \simeq \mathbb{Z}(t_1, \ldots t_n) = \mathbb{Z}[t_1, \ldots t_n, t_1^{-1}, \ldots t_n^{-1}]$$

the ring of Laurent polynomials on $t_1, \ldots t_n$ over $\mathbb{Z}$.

- Combined with the theorem this makes $K_T(Fl(n;\mathbb{C}))$ substantially more tractable.

Remark. Each $n!$-tuple of Laurent polynomials satisfying certain conditions in

$$\bigoplus_{S_n} \mathbb{Z}(t_1, \ldots t_n)$$

corresponds to exactly one element of $K_T^0(Fl(n;\mathbb{C}))$.

Two Bases

Line Bundle Basis

Below is a module basis for the set of line bundles on $Fl(3;\mathbb{C})$, visualized as a collection of injections to the fixed points of $R(T)$, with each point on a hexagon representing a fixed point.

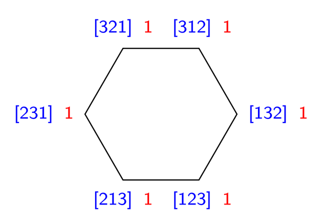


Figure: $1 \otimes 1$

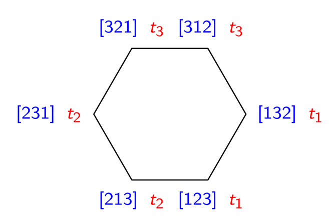


Figure: $1 \otimes t_1$

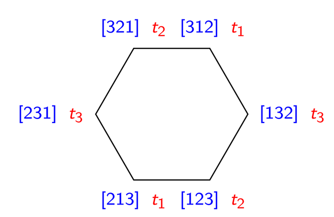


Figure: $1 \otimes t_2$

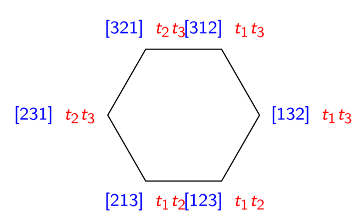


Figure: $1 \otimes t_1 t_2$

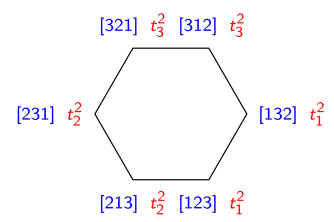


Figure: $1 \otimes t_1^2$

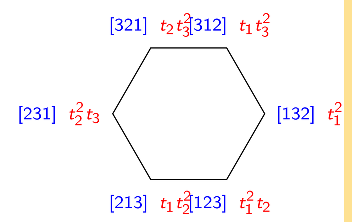


Figure: $1 \otimes t_1^2 t_2$

Schubert Basis

Below is a description of the Schubert basis as injections to the fixed points

[123]	[213]	[231]	[321]	[312]	[132]
1	1	1	1	1	1
0	$1 - t_2 t_1^{-1}$	$1 - t_2 t_1^{-1}$	$1 - t_3 t_1^{-1}$	$1 - t_3 t_1^{-1}$	0
0	0	$1 - t_3 t_1^{-1}$	$1 - t_3 t_1^{-1}$	$1 - t_3 t_2^{-1}$	$1 - t_3 t_1^{-1}$
0	0	$(1 - t_3 t_1^{-1})(1 - t_2 t_1^{-1})$	$(1 - t_3 t_1^{-1})(1 - t_2 t_1^{-1})$	0	0
0	0	0	$(1 - t_3 t_1^{-1})(1 - t_3 t_2^{-1})$	$(1 - t_3 t_1^{-1})(1 - t_3 t_2^{-1})$	0
0	0	0	$*$	0	0

$$* = (1 - t_3 t_1^{-1})(1 - t_3 t_2^{-1})(1 - t_2 t_1^{-1})$$

The Schubert Basis

A special set of *subvarieties* X_w of G/B that are indexed by permutations $w \in S_n$ and are called *Schubert varieties*.

$$X_w = \overline{BwB/B},$$

In terms of matrices for $G = Gl(n,\mathbb{C})$, we may represent a matrix as a permutation and then see which matrices are represented in X_w. If $w = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then

$$BwB/B = \begin{pmatrix} * & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad X_w = \begin{pmatrix} * & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $*$ can be any complex number and the matrices are chosen to have as many 0s as possible, using the equivalence by the right B action. The closure X_w of BwB/B also contains the identity matrix.

Structure sheaves on X_w are denoted $\mathcal{O}_{X_w}$. The corresponding classes $[\mathcal{O}_{X_w}]$ (each sheaf is taken up to an equivalence) form the *Schubert basis* of $K_T(G/B)$.

Change of Basis

The primary goal of this project was to find an explicit way to change from the Schubert Basis to the line bundle basis. While we were only able to obtain partial results, due to the computationally complex nature of the problem, we are in the process of finding more efficient ways to calculate this, which will hopefully generalize to higher dimension.

We were able to find some Schubert classes in the basis of line bundles.

$[\mathcal{O}_{X_{[123]}}] = 1 \otimes 1$

$[\mathcal{O}_{X_{[213]}}] = 1 \otimes 1 - t_1^{-1} \otimes t_1$

$[\mathcal{O}_{X_{[132]}}] = 1 \otimes 1 - (t_1 t_2)^{-1} \otimes t_1 t_2$

$[\mathcal{O}_{X_{[312]}}] = 1 \otimes 1 - t_3(t_1 + t_2) \otimes t_1 + t_3 \otimes t_1^2$

This translates between K_T-theory as an equivalence class of vector bundles, and K_T-theory as sheaves on Schubert varieties.

References

1. First item in a list
2. Second item in a list
3. Third item in a list
4. Fourth item in a list
5. Fifth item in a list
6. Sixth item in a list