On Changing the Basis of the T-Equivariant K-theory of Flag Manifolds

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Outline

Flag Manifolds

Definition

A **complete flag** *in* $\mathbb{C}^n$ *is a chain of vector subspaces*

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{C}^n$$

where the (complex) dimension of V_i *is* i *for all* $0 \leq i \leq n$.

For the purposes of this talk we'll just refer to a complete flag as a flag.

Definition

The (complete) **flag manifold** $Fl(n;\mathbb{C})$ *is the space of all complete flags in* $\mathbb{C}^n$.

Flag Manifolds

Remark

It is convenient to represent a flag as an invertible $n \times n$ matrix A where the subspace V_i is the span of the first i columns of X.

For examle, in $\mathbb{C}^3$ the matrix

$$g = \begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{2,1} & g_{2,2} & g_{2,3} \\ g_{3,1} & g_{3,2} & g_{3,3} \end{pmatrix}$$

represents the flag

$$\{0\} \subset \operatorname{span}\left\{ \begin{pmatrix} g_{1,1} \\ g_{2,1} \\ g_{3,1} \end{pmatrix} \right\} \subset \operatorname{span}\left\{ \begin{pmatrix} g_{1,1} \\ g_{2,1} \\ g_{3,1} \end{pmatrix}, \begin{pmatrix} g_{1,2} \\ g_{2,2} \\ g_{3,2} \end{pmatrix} \right\} \subset \operatorname{span}\left\{ \begin{pmatrix} g_{1,1} \\ g_{2,1} \\ g_{3,1} \end{pmatrix}, \begin{pmatrix} g_{1,2} \\ g_{2,2} \\ g_{3,2} \end{pmatrix}, \begin{pmatrix} g_{1,3} \\ g_{2,3} \\ g_{3,3} \end{pmatrix} \right\} = \mathbb{C}^3$$

Flag Manifolds

- Lots of different matrices represent the same flag.

Definition

Let $G = GL(n;\mathbb{C})$ be the **general linear group** *of invertible $n \times n$ complex valued matrices (under matrix multiplication).*

Definition

Let B denote the subgroup of upper triangular matrices.

Remark

For any $g \in G$ and for any $b \in B$, b and $g \cdot b$ have the same column spans.

- In other words, the span of the first i columns of g and $g \cdot b$ span the same subspace of $\mathbb{C}^n$.

Flag Manifolds

For example if we let

$$g = \begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{2,1} & g_{2,2} & g_{2,3} \\ g_{3,1} & g_{3,2} & g_{3,3} \end{pmatrix}, b = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ 0 & b_{2,2} & b_{2,3} \\ 0 & 0 & b_{3,3} \end{pmatrix}$$

then

$$g \cdot b = \begin{pmatrix} b_{1,1}g_{1,1} & b_{1,2}g_{1,1} + b_{2,2}g_{1,2} & b_{1,3}g_{1,1} + b_{2,3}g_{1,2} + b_{3,3}g_{1,3} \\ b_{1,1}g_{2,1} & b_{1,2}g_{2,1} + b_{2,2}g_{2,2} & b_{1,3}g_{2,1} + b_{2,3}g_{2,2} + b_{3,3}g_{2,3} \\ b_{1,1}g_{3,1} & b_{1,2}g_{3,1} + b_{2,2}g_{3,2} & b_{1,3}g_{3,1} + b_{2,3}g_{3,2} + b_{3,3}g_{3,3} \end{pmatrix}$$

Remark

The ith column of $g \cdot b$ is a $\mathbb{C}$-linear combination of the first i columns of g.

Flag Manifolds

Realizing that g and $g \cdot b$ have the same column spans demonstrates that:

Proposition

$$Fl(n; \mathbb{C}) \simeq G/B$$

- Abstract Algebra pop quiz: Is $Fl(n; \mathbb{C})$ a group?

A Group Action on Flag Manifolds

Definition

Let $T^n \simeq S^1 \times S^1 \times \cdots \times S^1$ be the **torus** *of dimension n.*

- We will realize T^{n-1} as a diagonal $n \times n$ matrix where $a_{j,j} = e^{i\theta_j}$ for $1 \leq j \leq n-1$, and $a_{n,n} = e^{-i(\theta_1+\cdots+\theta_{n-1})}$.
- One purpose as expressing $Fl(n;\mathbb{C})$ as G/B is realizing a left T^{n-1} action on $Fl(n;\mathbb{C})$.
- T acts on G/B by $t(gB) = (t \cdot g)B$.

Fixed Points of the T-action on the Flag manifold

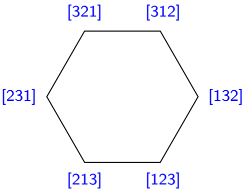
- The fixed points of this action are of particular interest.
- Which flags are fixed for all $t \in T$?

Remark

The set of T-fixed points of $Fl(n;\mathbb{C})$, denoted $Fl(n;\mathbb{C})^T$, is indexed by S_n, the symmetric group on $1,\ldots,n$. The fixed flags are realized as permutation matrices.

T^2 acts on $Fl(3;\mathbb{C})$; the fixed point set is indexed by

$$S_3 = \{[123],[213],[132],[231],[312],[321]\}.$$



Big scary K-theory definitions

Definition

For a smooth manifold X, denote by $K_T(X)$ the Grothendieck ring of T-equivariant vector bundles over X.

Proposition

$$K_T(\{\mathrm{pt}\}) \simeq R(T)$$

Theorem

(Steinberg and Kostant-Kumar)

$$K_T(Fl(n;\mathbb{C})) \simeq R(T) \otimes_{R(T)^{S_n}} R(T)$$

where $R(T)$ is the ring of representations of T.

Making sense of big scary definitions

Remark

$$R(T^n) \simeq \mathbb{Z}(t_1, \ldots t_n) = \mathbb{Z}[t_1, \ldots t_n, t_1^{-1}, \ldots t_n^{-1}],$$

the ring of Laurent polynomials on $t_1, \ldots t_n$ over $\mathbb{Z}$.

- Combined with the theorem this makes $K_T(Fl(n;\mathbb{C}))$ substantially more tractable.

Remark

Each $n!$-tuple of Laurent polynomials satisfying certain conditions in

$$\bigoplus_{S_n} \mathbb{Z}(t_1, \ldots t_n)$$

corresponds to exactly one element of $K_T^0(Fl(n;\mathbb{C}))$.

Injection to Fixed Points

In more formal terms this says:

Theorem

(Knutson-Rosu) The inclusion

$$Fl(n;\mathbb{C})^T \hookrightarrow Fl(n;\mathbb{C})$$

induces an injection on equivariant K-theory:

$$K_T(Fl(n;\mathbb{C})) \hookrightarrow K_T(Fl(n;\mathbb{C})^T) \simeq \bigoplus_{S_n} \mathbb{Z}(t_1, \ldots t_{n-1})$$

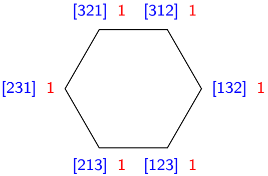
This makes it much easier to make computations in and understand the structure of $K_T(Fl(n;\mathbb{C}))$.

Line Bundle basis

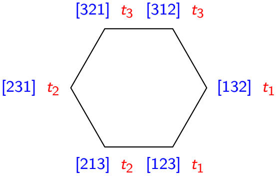
- Consider flags on $\mathbb{C}^3$.
- The ring $K_T(Fl(\mathbb{C}^3))$ is $R(T) \otimes_{R(T)^{S_n}} R(T)$ where

$$R(T) = K_T(pt) = \mathbb{Z}[t_1, t_2, t_1^{-1}, t_2^{-1}] \quad \text{(Laurent polynomials)}$$

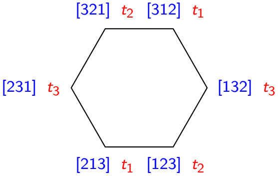
- There is a basis of $K_T(Fl(\mathbb{C}^n))$ as a module over $K_T(pt)$, represented by line bundles.
- Here is a module basis represented by line bundles: $1 \otimes 1$, $1 \otimes t_1$, $1 \otimes t_2$, $1 \otimes t_1 t_2$, $1 \otimes t_1 t_2$, $1 \otimes t_1^2$, $1 \otimes t_1^2 t_2$.



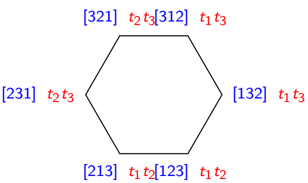
$1 \otimes 1$



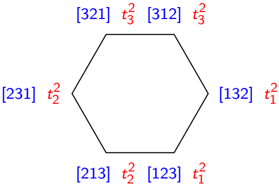
$1 \otimes t_1$



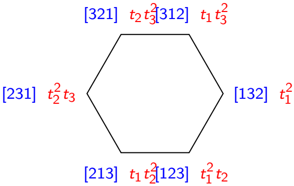
$1 \otimes t_2$



$1 \otimes t_1 t_2$



$1 \otimes t_1^2$



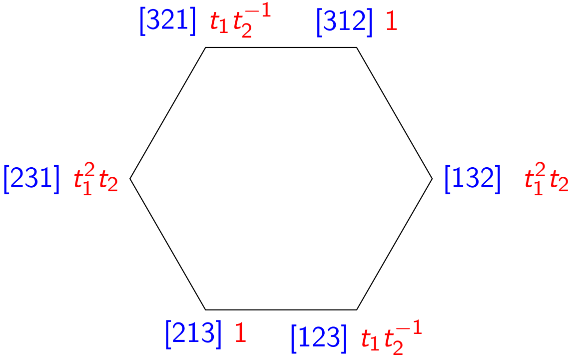
$1 \otimes t_1^2 t_2$

- Each $f(t_1, t_2) \otimes g(t_1, t_2)$ maps to $f(t_1, t_2)g(t_{\sigma(1)}, t_{\sigma(2)})$ on the fixed point associated to the permutation σ. Note: $t_3 = (t_1 t_2)^{-1}$.
- Any of these (or "linear" combinations thereof) can be restricted to the fixed points. When we do this, the variables on each fixed point correspond to permuting the indices of the variables to the right of the tensor product (using $t_3 = t_1^{-1} t_2^{-1}$), and then multiplying each point by the variable to the left of the tensor product.

We can represent this using a hexagon. We start by placing the fixed point corresponding to the identity permutation in the bottom right. Then, we can pick three directions, and associate these with the transpositions $(12), (23)$, and (13). We can then left multiply by these to label the remaining points of the hexagon. We restrict

$$t_1 \otimes t_2^{-1}$$

to the fixed points.



Schubert basis

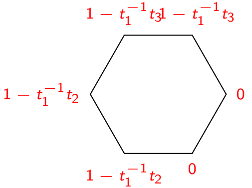
We don't have Schubert basis in terms of $R(T) \otimes_{R(T)^W} R(T)$. That is one reason to express these classes as a linear combination of line bundles.

[123]	[213]	[231]	[321]	[312]	[132]
1	1	1	1	1	1
0	$1 - t_2 t_1^{-1}$	$1 - t_2 t_1^{-1}$	$1 - t_3 t_1^{-1}$	$1 - t_3 t_1^{-1}$	0
0	0	$1 - t_3 t_1^{-1}$	$1 - t_3 t_1^{-1}$	$1 - t_3 t_2^{-1}$	$1 - t_3 t_1^{-1}$
0	0	$(1 - t_3 t_1^{-1})(1 - t_2 t_1^{-1})$	$(1 - t_3 t_1^{-1})(1 - t_2 t_1^{-1})$	0	0
0	0	0	$(1 - t_3 t_1^{-1})(1 - t_3 t_2^{-1})$	$(1 - t_3 t_1^{-1})(1 - t_3 t_2^{-1})$	0
0	0	0	*	0	0

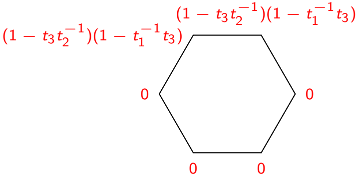
$$* = (1 - t_3 t_1^{-1})(1 - t_3 t_2^{-1})(1 - t_2 t_1^{-1})$$

Some Schubert classes, by restrictions

Each Schubert class is indexed by a single permutation. Here are two of them:



$S_{[132]}$



$S_{[312]}$

Some Code in Python

- Finding the way to write a K-Theory class in terms of the line bundle basis got tedious, so we wrote some code in SymPy that treats this problem like a standard, linear algebra problem of solving $A\vec{x} = \vec{b}$.
- This code made some of these easier, although it could not handle solving all such problems.

```python
from sympy import *

t1 = Symbol("t1")
t2 = Symbol("t2")
t3 = Symbol("t3")
t3 = (t1*t2)**(-1)

###(id, [213], [231], [321], [312], [132])
### The line bundle bases
o1 = [1,1,1,1,1,1]
ot1 = [t1,t2,t2,t3,t3,t1]
ot2 = [t2,t1,t3,t2,t1,t3]
ot1t2 = [t1*t2,t1*t2,t2*t3,t2*t1**(-1),t1*(t2**(-1)),t2**(-1)]
ot12 = [t1**2,t2**2,t2**2,t3**2,t3**2,t1**2]
ot12t2= [t2*(t1**2),t2*(t1**2),t2*(t1**(-1)),(t1**2)*(t2**(-1)),t3*(t2**(-1)),t1*(t2**(-1))]

### Compiles the bases into a matrix
M = Matrix([o1,ot1,ot2,ot1t2,ot12,ot12t2])
Mt = M.transpose()


### Writes an injection to the fixed points in terms of the basis
class basis:
    def __init__(self,lis):
        if len(lis) != 6:
            raise IndexError
            return
        self.augmented = Mt.col_insert(6,Matrix(lis))
        self.result = self.augmented.rref()[0]
```

Expressing inverses of line bundles in module basis of line bundles

Each element of the line bundle basis has a corresponding inverse bundle, (e.g. $1 \otimes t_1$ has the inverse bundle $1 \otimes t_1^{-1}$).
Conveniently, these can also be expressed in terms of the original basis.

$1 \otimes 1 = 1 \otimes 1$

$1 \otimes t_1^{-1} = (t_1^{-1} + t_2^{-1} + t_3^{-1}) \otimes 1 - (t_1 + t_2 + t_3) \otimes t_1 + 1 \otimes t_1^2$

$1 \otimes t_2^{-1} = (t_1 + t_2 + t_3) \otimes t_1 - 1 \otimes t_1^2 - 1 \otimes t_2$

$1 \otimes t_1^{-1} t_2^{-1} = (t_1 + t_2 + t_3) \otimes 1 - 1 \otimes t_1 - 1 \otimes t_2$

$1 \otimes t_1^{-2} = (t_1 + t_2 + t_3 + t_1^{-2} + t_2^{-2} + t_3^{-2}) \otimes 1 - (t_1 t_3^{-1} + t_2 t_3^{-1} + t_1 t_2^{-1} + t_2 t_1^{-1} + t_1 + t_2 + 2) \otimes t_1 + (t_1^{-1} + t_2^{-1} + t_3^{-1}) \otimes t_1 t_2$

Toward a change of basis

We were able to find some Schubert classes in the basis of line bundles.
$S_{123} = 1 \otimes 1$
$S_{213} = 1 \otimes 1 - t_1^{-1} \otimes t_1$
$S_{132} = 1 \otimes 1 - (t_1 t_2)^{-1} \otimes t_1 t_2$
$S_{312} = 1 \otimes 1 - t_3(t_1 + t_2) \otimes t_1 + t_3 \otimes t_1^2$
This translates between K_T-theory as an equivalence class of vector bundles, and K_T-theory as sheaves on some subvarieties.