Fun with Buns!

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Motivation

A common question in topology is how to characterize certain structures that lie "above" a given topological space, say M. For example, given that M is well connected, how can we describe covering spaces over M? It turns out that the data of a covering space is equivalently the data of an action of the fundamental group at any point of M (as M is well connected this is consistent) on a set.

That this topological question is equivalent to an algebraic one is quite amazing. If we look at a more natural object called the fundamental groupoid, we can state this as an equivalence of categories $\mathrm{Fun}(\Pi_1(M), \mathrm{Set}) \simeq Cov/M$

This equivalence is due to the unique path lifting property of covering spaces. If we look at it from the algebraic side, what happens if we change the target category?

The Relevant Categories

- $\Pi_1(M)$ is the fundamental groupoid of M, with the set of objects just the set M itself, and with homotopy classes of paths as morphisms.
- $\mathsf{Vect}_{\mathbb{R}}^{\text{f.d.}}$ is the category of finite dimensional vector spaces and linear maps between them.
- Given two categories $\mathcal{C}$ and $\mathcal{D}$, $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ is the category of functors between them and natural transformations between such functors.
- $\mathsf{Bun}_\flat(\mathsf{Vect}_{\mathbb{R}}^{\text{f.d.}}, M)$ is the category of vector bundles with homotopy invariant parallel transport, and vector bundle homomorphisms which are "equivariant" with respect to the transport.

Interlude: Universal Covers

Unpacking the equivalence mentioned earlier between functors from the fundamental groupoid into sets and covering spaces over M, there is a functor $\int_M : \mathrm{Fun}(\Pi_1(M), \mathrm{Set}) \to Cov/M$ that takes a functor $F : \Pi_1(M) \to \mathrm{Set}$ to a covering space $\int_M F$ which as a set is just the disjoint union of all the sets Fx.

We will assume from now on that we are working with a connected manifold M, with a fixed basepoint x and fixed good cover $\{U_\alpha \hookrightarrow M\}_{\alpha \in A}$. There is a canonical functor $\mathrm{Hom}(x, -)$ that takes every point y to the set of homotopy classes of paths ending at y, and every path into a post-composition function. If you apply $\int_M$ to this functor and check the automorphism group of the cover, you will find that this is in fact the universal cover of M, and in fact is a principal $\pi_1(M, x)$-bundle as well.

From Functors to Bundles

In the case of $\mathrm{Fun}(\Pi_1(M), \mathrm{Vect}_{\mathbb{R}}^{\mathrm{f.d.}})$, where we have replaced the target category, what kind of natural object can we create? Remember that a functor from the fundamental groupoid to sets was essentially an action of the group $\pi_1(M, x)$ on a set.

In a similar vein, a functor $F : \Pi_1(M) \to \mathrm{Vect}_{\mathbb{R}}^{\mathrm{f.d.}}$ is kind of like a representation of $\pi_1(M, x)$ on Fx. Given a principal $G-$bundle P and finite dimensional representation V of G, we can create an associated vector bundle by forming the quotient of the product $P \times V/\sim$, a kind of colimit. Doing something similar to this, the fiber above y is the set of equivalence classes $[\gamma : x \to y, \vec{v} \in Fy]$ equipped with a vector space structure. A natural transformation between functors F and H in $\mathrm{Fun}(\Pi_1(M), \mathrm{Vect}_{\mathbb{R}}^{\mathrm{f.d.}})$ is sent to the map that at each fiber is just the component of the natural transformation at the basepoint of the fiber.

From Bundles to Functors

Given a vector bundle over M, we can form a function that takes each point y of M and returns the vector space E_y. However, there isn't really a canonical choice of isomorphism between those vector spaces, unless your bundle is explicitly trivial.

The notion of a linear connection on the bundle helps to alleviate this problem, as we can give a path $\gamma : y \to z$ in M the job of "transporting" different vectors along to vectors above the endpoints of the path, providing an isomorphism between the vector spaces above the endpoints. However, in general this is dependent on more than just the homotopy class of the path taken. This is because the connection has a curvature associated with it, that you can actually compute using the transport associated with the connection. In the case of a flat connection, the parallel transport is actually homotopy invariant, and assembles into a functor $\Pi_1(M) \to \mathrm{Vect}_{\mathbb{R}}^{\mathrm{f.d.}}$!

Functors = Bundles + Stuff!

We can equip our bundles-from-functors with a parallel transport, using the fact that the universal cover has a canonical flat connection (and the parallel transport over that is really just the unique path lifting property) and pulling it back to the vector bundle. Even more interestingly, our natural transformations respect that parallel transport! Changing perspective, vector bundle homomorphisms that respect parallel transport can also be made into natural transformations between the parallel transport functors of the respective bundles. These assignments are both functorial!

Now, given functors pointing the opposite way, we would be remiss to not ask the obvious question of whether or not they are inverse to each other. However, this isn't really fair to ask of these functors. As long as they are *essentially* inverses of each other, we should consider the two categories the same. So, we can construct natural isomorphisms between the composition of the functors and the respective identities of the categories.