Wave Fronts in DTDS Population Models

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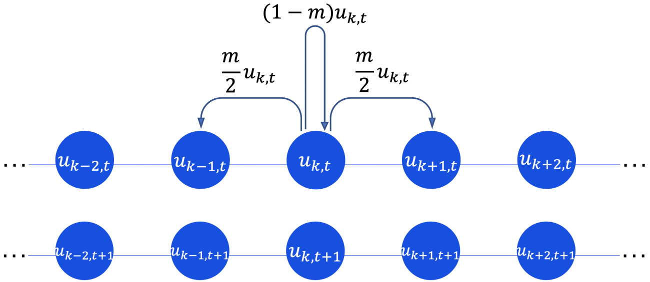
The Model

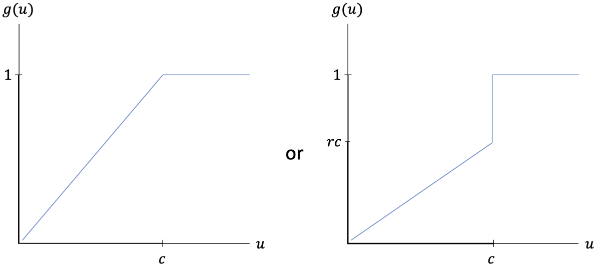
- Infinite one-dimensional lattice of nodes.
- Each node has a continuous population value, $u_{k,t}$.
- Each generation, population migrates, then grows.



- $u_{k,t+1} = g\left(\frac{m}{2}u_{k-1,t} + (1-m)u_{k,t} + \frac{m}{2}u_{k+1,t}\right)$

Growth Function

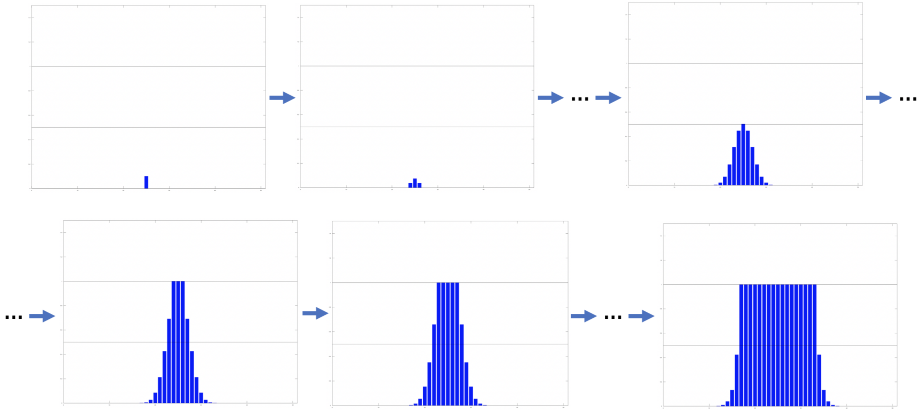
- We focus on a capped linear growth function.
- $g(u) = \begin{cases} ru, & r < c \\ 1, & r \geq c \end{cases}$ And we impose $rc \leq 1$.



Wave Fronts and Wave Speed

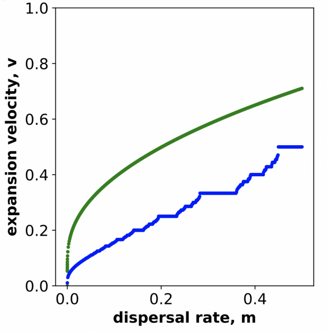
- Initial condition:
  $P = (\ldots, u_{-2,0}, u_{-1,0}, u_{0,0}, u_{1,0}, u_{2,0}, \ldots) = (\ldots, 0, 0, u_0, 0, 0, \ldots)$
- Wave front: $d(t) = \max_{j \in \mathbb{Z}} \{u_{j,t} = 1\}$
- Wave speed: $s = \lim_{t \to \infty} \frac{d(t)}{t}$
- Some waves have **rational** speeds $s = \frac{p}{q}$.
- After q generations, the wave has shifted p nodes to the right.
- $u_{k+p,t+q} = u_{k,t}$.

Example Simulation



Speed Locking

- Fixing r and c and varying m produces interesting results [1].



Green: $r = 1.1, c = 0.4$

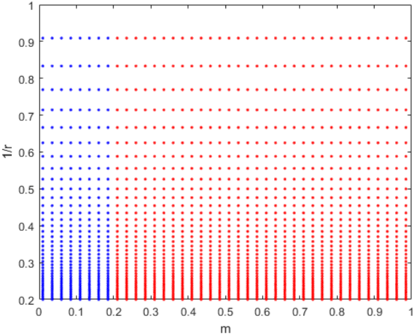
Blue: $r = 1.1, c = 0.5$

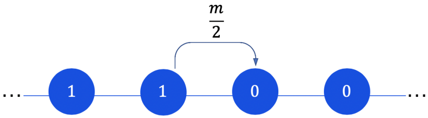
- "Devil's staircase" structure occurs with plateaus at rational numbers.

- This fractal pattern only occurs for some parameter choices.

Speed 1 vs. Speed Less Than 1

- Parameter Restrictions:
  - $0 < m < 1$
  - $0 < c \leq 1$
  - $r > 1$
  - $rc \leq 1$
- Blue: Speed less than 1
- Red: Speed 1
- For this graph, $c = \frac{1}{10}$.
- Speed less than 1 only occurs when $m < 2c$.





Speed 1 Fronts

Types of Speed 1 Waves:

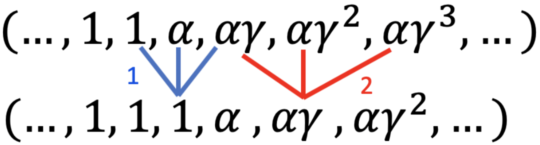
- $(1,0)$ Fronts:
  $P(t) \quad\;\; = (\dots,1,1,0,0,0,\dots)$
  $P(t+1) = (\dots,1,1,1,0,0,\dots)$

- $(1,\alpha,0)$ Fronts:
  $P(t) \quad\;\; = (\dots,1,1,\alpha,0,0,\dots)$
  $P(t+1) = (\dots,1,1,1,\alpha,0,\dots)$

- $(1,\alpha,\alpha\gamma)$ Fronts (Exponential Decay):
  $P(t) \quad\;\; = (\dots,1,1,\alpha,\alpha\gamma,\alpha\gamma^2,\dots)$
  $P(t+1) = (\dots,1,1,1,\ \alpha\ ,\ \alpha\gamma\ ,\dots)$

Speed 1 Approach

Example: $(1, \alpha, \alpha\gamma)$ front analysis

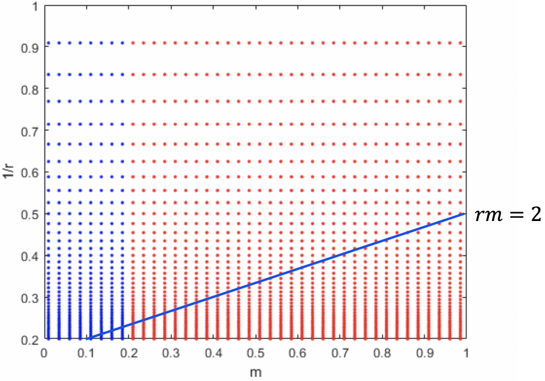


Define $M(x, y, z) = \frac{m}{2}x + (1 - m)y + \frac{m}{2}z$.
1: $M(1, \alpha, \alpha\gamma) \geq c$
2: $M(\alpha\gamma, \alpha\gamma^2, \alpha\gamma^3) < c$
$rM(\alpha\gamma, \alpha\gamma^2, \alpha\gamma^3) = \alpha\gamma$

- A node's population is determined by its three "parents."
- Using the migration and growth equations, we can solve for **constraints** on our parameters.

Speed 1 Findings

- We were unable to find any $(1, \alpha, \alpha\gamma)$ fronts with speed 1 in practice.
- We have shown $(1, \alpha, 0)$ fronts only exist on the line $rm = 2$ (or $\frac{1}{r} = \frac{m}{2}$).
- Fronts off of this line degenerate to $(1, 0)$ fronts.
- If the current state is $(\ldots, 1, 1, \alpha, 0, 0, \ldots)$, then the node to the right of α gets mapped to $\frac{rm}{2}\alpha$.



Speed $\frac{1}{2}$ Approach

Example: $(1, \alpha, \alpha\gamma)$ front analysis

$$P(t) \quad = (\dots, 1, 1, \alpha, \alpha\gamma, \alpha\gamma^2, \alpha\gamma^3, \alpha\gamma^4, \alpha\gamma^5, \alpha\gamma^6, \dots)$$

$$P(t+1) = (\dots, 1, 1, \beta, \beta\gamma, \beta\gamma^2, \beta\gamma^3, \beta\gamma^4, \beta\gamma^5, \beta\gamma^6, \dots)$$

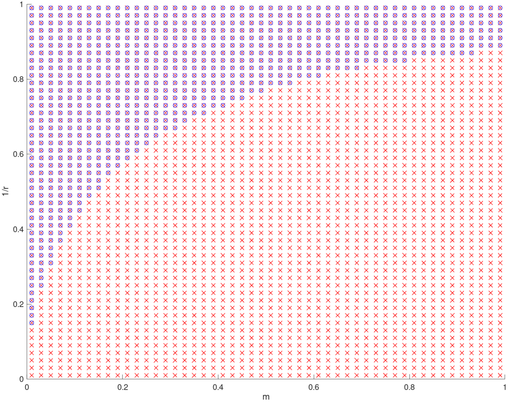
$$P(t+2) = (\dots, 1, 1, 1, \alpha, \alpha\gamma, \alpha\gamma^2, \alpha\gamma^3, \alpha\gamma^4, \alpha\gamma^5, \dots)$$

Node $\alpha\gamma$ is determined by its five "grandparents."
We can describe the function F such that $F(\alpha, \alpha\gamma, \alpha\gamma^2, \alpha\gamma^3, \alpha\gamma^4) = \alpha\gamma$.

- We have multiple layers; we want to relate $P(t+2)$ back to $P(t)$.
- This gives us a quartic polynomial in γ with coefficients determined by r and m.
- We require that γ is real and $0 < \gamma < 1$.

Speed $\frac{1}{2}$ Roots

- X: complex root
- O: real root in $(0, 1)$
- The region where exponential decay speed $\frac{1}{2}$ fronts exist is a subset of this region.
- We will continue to narrow the boundaries of this region.
- In practice, we can find $(1, \gamma, \gamma^2)$ fronts.



Speed $\frac{1}{n}$ Fronts

- Unfortunately, speed $\frac{1}{n}$ fronts cannot be pure exponential decay fronts.
- We must use a generalized wave front that **approaches** exponential decay.
- Speed $\frac{1}{n}$ fronts:

$$
\begin{aligned}
P(t) &= (\dots, 1, 1, \alpha_1, \alpha_2, \alpha_3, \dots) \\
P(t+1) &= (\dots, 1, 1, \beta_1, \beta_2, \beta_3, \dots) \\
\dots \quad & \dots \qquad \dots \\
P(t+n-1) &= (\dots, 1, 1, \eta_1, \eta_2, \eta_3, \dots) \\
P(t+n) &= (\dots, 1, 1, \ 1, \alpha_1, \alpha_2, \dots)
\end{aligned}
$$

- To be quasi-exponential, we add the restrictions

$$\lim_{m\to\infty} \frac{\alpha_{m+1}}{\alpha_m} = \lim_{m\to\infty} \frac{\beta_{m+1}}{\beta_m} = \cdots = \lim_{m\to\infty} \frac{\eta_{m+1}}{\eta_m} = \gamma$$

Speed $\frac{1}{3}$ Fronts

$$(\dots, 1, 1, \alpha, \alpha\gamma, \alpha\gamma^2, \alpha\gamma^3, \alpha\gamma^4, \alpha\gamma^5, \alpha\gamma^6, \dots)$$

$$(\dots, 1, 1, \beta, \beta\gamma, \beta\gamma^2, \beta\gamma^3, \beta\gamma^4, \beta\gamma^5, \beta\gamma^6, \dots)$$

$$(\dots, 1, 1, \delta, \delta\gamma, \delta\gamma^2, \delta\gamma^3, \delta\gamma^4, \delta\gamma^5, \delta\gamma^6 \dots)$$

$$(\dots, 1, 1, 1, \alpha, \alpha\gamma, \alpha\gamma^2, \alpha\gamma^3, \alpha\gamma^4, \alpha\gamma^5, \dots)$$

- We are currently working on speed $\frac{1}{3}$ fronts.
- Now, the values of γ are the roots of a polynomial of degree 6.
- $\gamma^2 = (\frac{rm}{2} + r(1-m)\gamma + \frac{rm}{2}\gamma^2)^3$
- We are working on generalizing this approach to all $\frac{1}{n}$ speeds.
- $\gamma^{n-1} = (\frac{rm}{2} + r(1-m)\gamma + \frac{rm}{2}\gamma^2)^n$

Future Work

Main focus:

- What conditions cause each type of speed $\frac{1}{n}$ front?
- How can we express general speed $\frac{p}{q}$ fronts?

Additional subjects:

- What do **irrational** speed waves look like, and when do they occur?
- What parameter ranges cause the **speed locking** effect?

References

[1] A.B. George, K.S. Korolev, S. Matin, C. Wang, Pinned, locked, pushed, and pulled traveling waves in structured environments, *Theoretical Population Biology* **127**, 102-109 (2019)