Wave Fronts in DTDS Population Models

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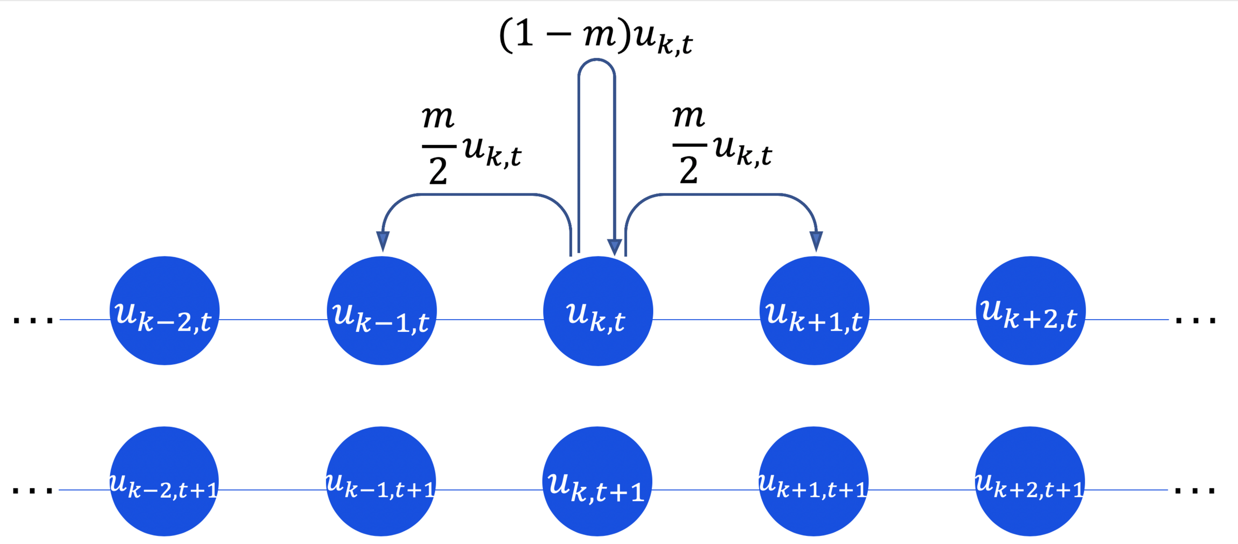
Introduction

1. The aim of this research is to study the occurrence of "wave fronts" in population growth models.
2. In these models, both time and space can be treated as discrete or continuous.
3. The model we focus on is a discrete-time, discrete-space (DTDS) model with a linear growth rate.

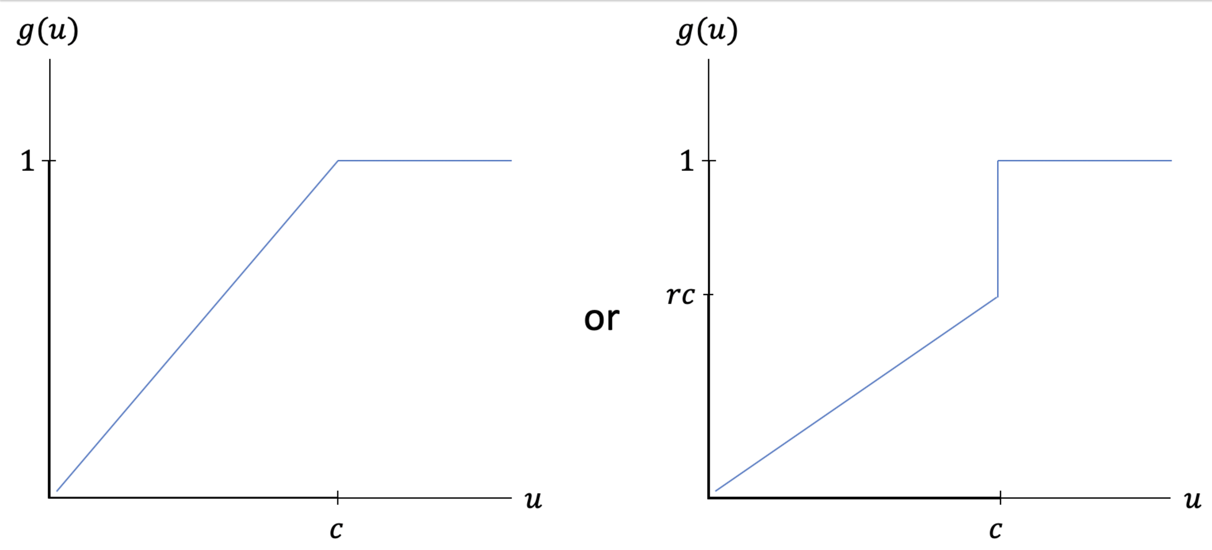
Description of the Model

1. Population sites are nodes on an infinite one-dimensional lattice.
2. Each node has a continuous population value $u_{k,t} \in [0,1]$, where k denotes the lattice site, and t denotes the generation number.
3. Each generation, the populations **migrate** and then **grow**, according to the function $u_{k,t+1} = g(\frac{m}{2}u_{k-1,t} + (1-m)u_{k,t} + \frac{m}{2}u_{k+1,t})$, where m is the **migration rate**, and g is the **growth function**.
4. The migration rate $m \in [0,1]$ is the proportion of $u_{k,t}$ that migrates away from the node.



Growth Function

1. We focus on a capped linear growth function, given by $g(u) = \begin{cases} ru, & r < c \\ 1, & r \geq c \end{cases}$, and we impose $rc < 1$.
2. The value r is the **reproduction rate** and, for our purposes, $r \in (1, \infty)$.
3. The value $c \in [0,1]$ is the **critical size**, after which a node's population is at capacity.



Waves

Wave Fronts

1. The **wave front** is the most rightward node with a population of 1. The wave front $d(t)$ is given by $d(t) = \max_{j \in \mathbb{Z}}\{u_{j,t} = 1\}$.
2. Let $P(t) = (\dots, u_{-2,t}, u_{-1,t}, u_{0,t}, u_{1,t}, u_{2,t}, \dots)$ be the state of the linear lattice at time t.
3. We consider waves resulting from the initial condition $P(0) = (\dots, 0, 0, u_0, 0, 0, \dots)$ for some $u_0 \in [0,1]$.

Wave Speeds

1. A wave's **speed** is defined as the distance the wave has traveled (i.e. $d(t)$) divided by the time it took to get there (i.e. t).
2. Since we wish to ignore initial turbulence, we consider the limit of this ratio as t approaches infinity.
3. Thus the speed of the wave is defined as $s = \lim_{t \to \infty} \frac{d(t)}{t}$.

Analyzing Speeds

1. An important category of waves is the class of waves with perfectly **rational** speeds.
2. A wave with rational speed $\frac{p}{q}$ can have the property that after q generations, the wave is identical, but expanded outward by p units in both directions.
3. In other words, all the nodes to the right of $u_{0,t}$ look like they have shifted right by p units after q generations.

Waves of Speed 1

1. Speed 1 fronts (i.e. waves that are shifted right by 1 unit each generation) are a natural starting point.
2. We were able to find a region within r and m parameter space where speed 1 fronts exist.
3. This was done by deriving a set of constraints that arise from the equations for migration/reproduction.
4. We found that there are two main types of speed 1 fronts, which we call $(1,0)$ fronts and $(1,\alpha,0)$ fronts.
5. We also examined $(1,\alpha,\alpha\gamma)$ fronts, but could not find any in simulations.

Waves of Speed Less Than 1

1. Unfortunately, general speed $\frac{p}{q}$ fronts don't have some of the nice properties of speed 1 fronts. These waves require an **infinite decaying tail** to be able to travel at a stable speed.
2. Unlike in the speed 1 case, we cannot easily derive parameter constraints by looking at the evolution of individual nodes.
3. Instead we derived an expression for the mapping from $P(t)$ to $P(t+q)$.
4. So far, we have only been able to theoretically analyze speed $\frac{1}{2}$ waves. We have found $(1,\alpha,\alpha\gamma)$ waves, but only for $\alpha = 1$.

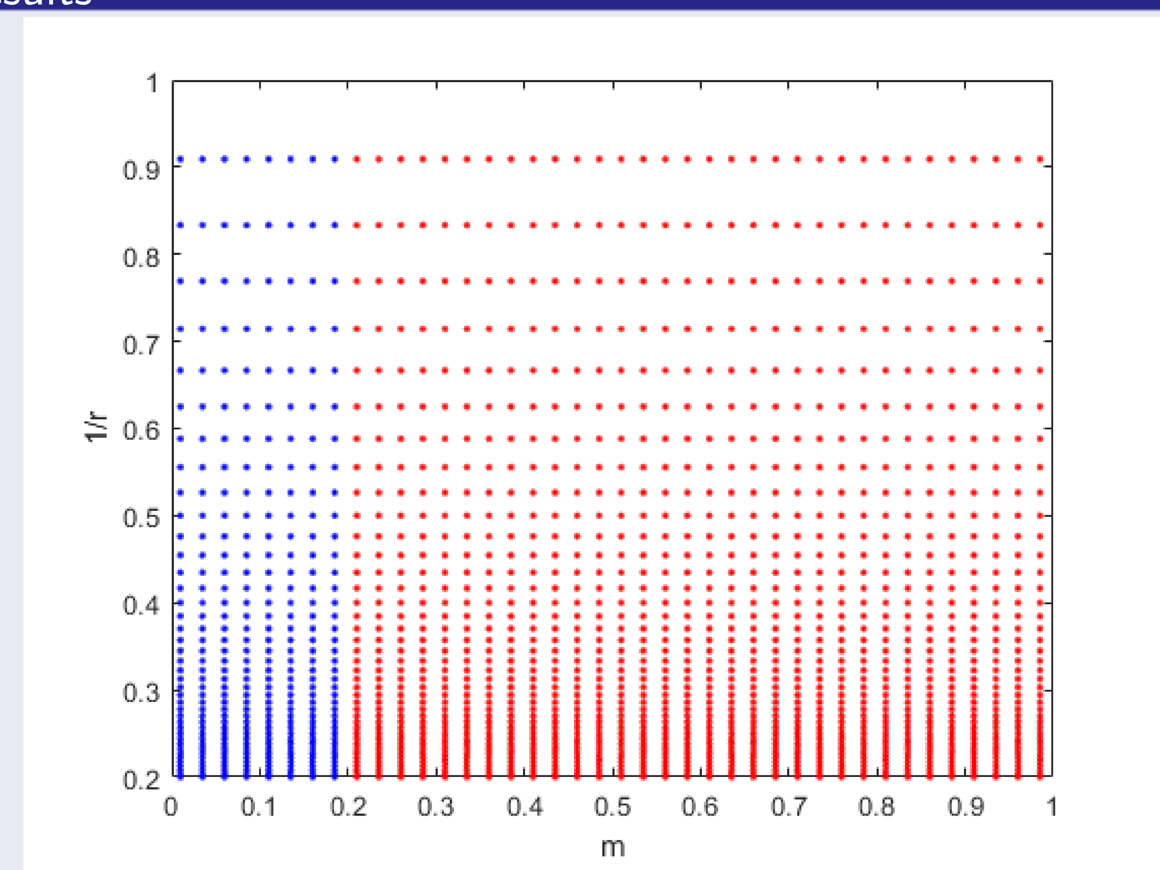
Types of Waves

1. Speed 1, $(1,0)$ fronts:
   $t: \quad (\dots, 1, 1, 0, 0, \dots)$
   $t+1: (\dots, 1, 1, 1, 0, \dots)$
2. Speed 1, $(1,\alpha,0)$ fronts:
   $t: \quad (\dots, 1, 1, \alpha, 0, 0, \dots)$
   $t+1: (\dots, 1, 1, 1, \alpha, 0, \dots)$
3. Speed $\frac{1}{2}$, $(1,\alpha,\alpha\gamma)$ fronts:
   $t: \quad (\dots, 1, 1, \alpha, \alpha\gamma, \alpha\gamma^2, \alpha\gamma^3, \dots)$
   $t+1: (\dots, 1, 1, \beta, \beta\gamma, \beta\gamma^2, \beta\gamma^3, \dots)$
   $t+2: (\dots, 1, 1, 1, \alpha, \alpha\gamma, \alpha\gamma^2, \dots)$
4. Speed $\frac{1}{n}$, generalized fronts:
   $t: \quad (\dots, 1, 1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots)$
   $t+1: \quad (\dots, 1, 1, \beta_1, \beta_2, \beta_3, \beta_4, \dots)$
   $\dots$
   $t+n-1: (\dots, 1, 1, \eta_1, \eta_2, \eta_3, \eta_4, \dots)$
   $t+n: \quad (\dots, 1, 1, 1, \alpha_1, \alpha_2, \alpha_3, \dots)$
5. These mapping requirements are used to derive constraints for the different kinds of waves.

Speed 1 Parameter Ranges

1. For speed 1 waves to arise from our initial condition, the following conditions must be met: $0 < m < 1$, $0 < rc < 1$, and $r > 1$.
2. If we also impose $rm = 2$, then we can achieve $(1,\alpha,0)$ fronts.
3. Otherwise, the wave will degenerate to a $(1,0)$ front.

Results



1. This graph demonstrates how wave speed depends on r and m at a fixed c value ($\frac{1}{10}$ in this example).
2. Red dots indicate speed 1. Blue dots indicate speed less than 1.
3. Note that waves of speed more than 1 are not attainable from our initial condition.
4. It appears that speeds less than 1 occur when $m < 2c$, which is related to the conditions we derived for $(1,\alpha,0)$ fronts.

Future Work

1. We are actively working on speed $\frac{1}{3}$ fronts, which have a similar structure to speed $\frac{1}{2}$.
2. However, it appears that speed $\frac{1}{n}$ do not follow a perfect exponential decay for $n > 2$, rather they only approach exponential decay as you move outwards.
3. Once we can understand speed $\frac{1}{3}$, we hope to generalize our findings to speed $\frac{1}{n}$ fronts
4. After we understand speed $\frac{1}{n}$, we hope to extend our analysis to all rational speeds $\frac{p}{q}$.
5. Lastly, we wish to study **irrational** speed fronts and determine if they exist.

References

A.B. George, K.S. Korolev, S. Matin, C. Wang, Pinned, locked, pushed, and pulled traveling waves in structured environments, *Theoretical Population Biology* **127**, 102-109 (2019)