Thurston Geometries in Virtual Reality

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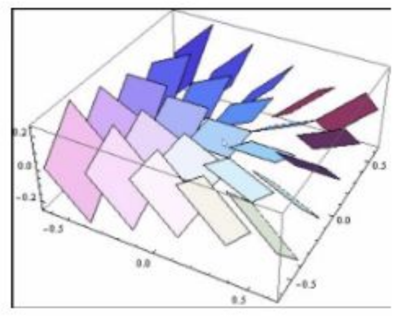
Geodesics in Nil

- The group law for multiplying vectors in Nil:
  $\{x_1, y_1, z_1\} \cdot \{x_2, y_2, z_2\} = \{x_1 + x_2, y_1 + y_2 + \alpha(x_1 z_2 - x_2 z_1), z_1 + z_2\}$
  *where α is a constant
- One direction stands out along which regular euclidean motion is observed, in our case the Y-axis
- Any angle off the Y-axis will result in spiral-like geodesics, and the further away you are from the origin the more intense this effect becomes



- We had a little help from Marenich who utilized covariant derivatives of the Riemannian connection of the left-invariant metric

Then the equation of a geodesic $\nabla_{\dot{c}(t)}\dot{c}(t) \equiv 0$ and our table of covariant derivatives (1) give:

$$\sum_{i=1}^{n}(\alpha_i'(t) + 2\gamma\beta_i(t))X_i(t) + \sum_{i=1}^{n}(\beta_i'(t) - 2\gamma\alpha_i(t))Y_i(t) + \gamma'(t)T = 0.$$

- From there we calculate an equation for geodesics from the origin, and use the multiplication rule to find geodesics for any other arbitrary point

Theorem 1. *Geodesic lines issuing from zero* **0** *in the Heisenberg group* H^{2n+1} *satisfy to the following equations:*

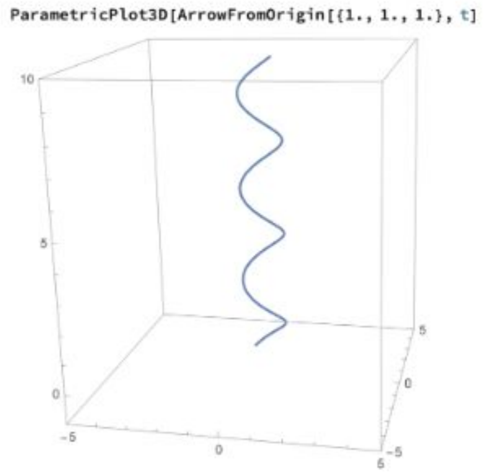
$$\begin{cases} x_i(t) = \frac{r_i}{2\gamma}(\sin(2\gamma t + \phi_i) - \sin(\phi_i)) \\ y_i(t) = \frac{r_i}{2\gamma}(\cos(\phi_i) - \cos(2\gamma t + \phi_i)) \\ z(t) = \frac{1+\gamma^2}{2\gamma}t - \frac{1-\gamma^2}{4\gamma^2}\sin(2\gamma t) \end{cases} \quad (11)$$

Citations

- Hart, V., Hawksley, A., Segerman, H., Bosch, M. T. (2015, July 21). Hypernom: Mapping VR Headset Orientation to $\mathbb{S}^3$. Retrieved from `https://arxiv.org/abs/1507.05707`
- Marenich, Valery. (1997). Geodesics in Heisenberg Groups. Geometriae Dedicata. 66. 175-185. 10.1023/A:1004916117293.
- Segerman, H. (n.d.). Retrieved January 25, 2020, from `http://www.segerman.org/XR.html`
- Weeks, J. (n.d.). Retrieved January 25, 2020, from `http://www.geometrygames.org/CurvedSpaces`

Arrow Consistency Property

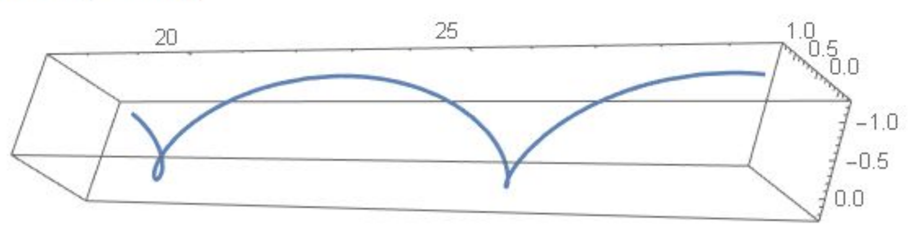
- First we fire arrow1 from the origin in some direction at some time t, giving us an equation for the path of the arrow



- We then fire arrow2 from wherever arrow1 is at some later time t0, in the same direction as arrow1
- We measure the distance between arrow2 at time t and arrow1 at time t+t0 to ensure that they are in the same place and following the same path

We used Mathematica to show the paths of the arrows in some direction after some initial time, denoted t_0

```
Show[
 ParametricPlot3D[arrow1[t + t0], {t, 0, 10}],
 ParametricPlot3D[arrow2[t, 1], {t, 0, 10}],
 PlotRange → All]
```



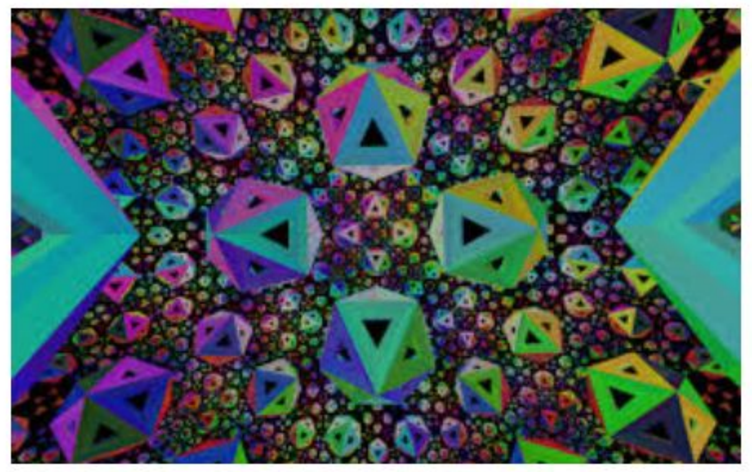
Background

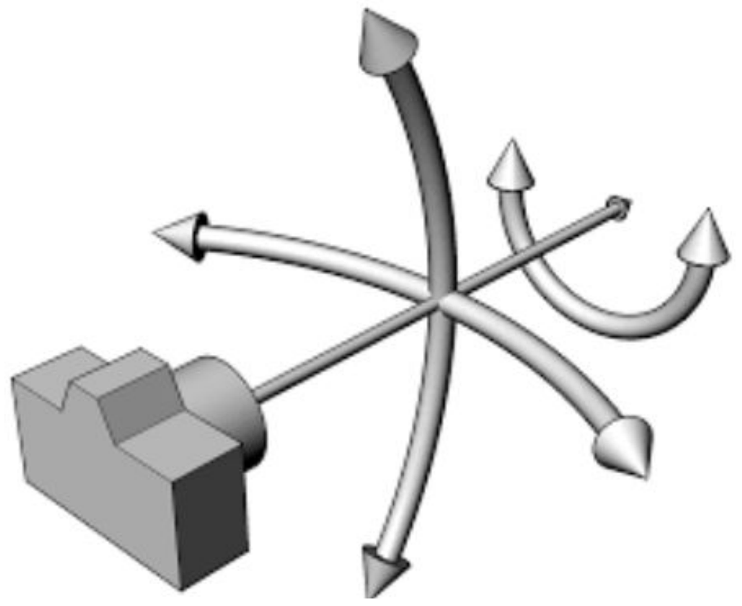
Objective: Create an archery simulation to visualize different geometries

- Different geometries will have different geodesics (straight lines)
- Path of fired arrow will follow geodesics in different geometries
- VR headset capabilities give a more immersive experience to gain a sense of what it would be like inside certain geometries
- Multiplayer functionality where geometrical features are passed over the network

Other Works

- We were inspired by other VR projects such as Hypernom that use techniques like ray tracing to visualize objects in different geometries
- The goal is that the user can explore geometries using VR by moving their head around in every possible orientation





Future Plans

1. Expand VR application
   1. Multiplayer functionality
   2. Mapping vertices of objects as well
2. Implement more geometries
   1. Hyperbolic
   2. Spherical
3. Create mini-games that challenge understanding of the geometry
   1. Use arrow destination for movement
   2. Laser path preview of the arrow
   3. Hitting a target placed somewhere