Change of Basis in the Equivariant K-theory of Flag Manifolds

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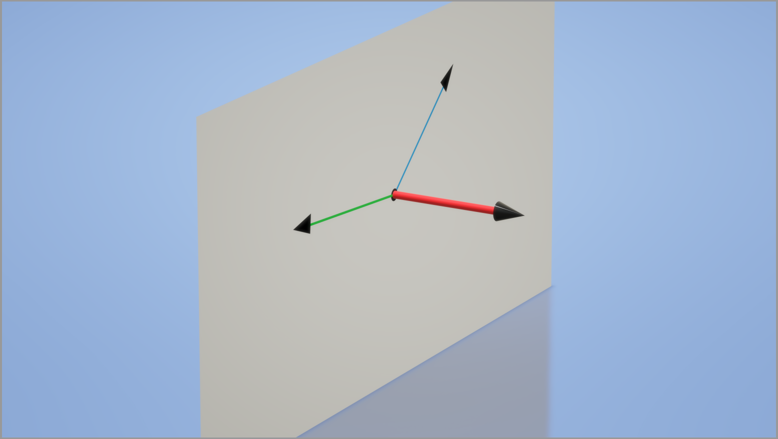
December 12, 2018

What is a Flag?

- Fix a vector space $\mathbb{C}^n$ for some nonnegative integer n. A (complete) **flag** on $\mathbb{C}^n$ is a maximal chain of vector subspaces, i.e. a chain $\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n$ with each V_i having dimension i.
- The set of all flags of $\mathbb{C}^n$ is denoted by $\mathrm{Fl(n, \mathbb{C})}$.

Example of a Flag



A Natural Torus Action on $\mathrm{Fl}(\mathrm{n}, \mathbb{C})$

- $\mathrm{Fl}(\mathrm{n}, \mathbb{C}) \cong G/B$ where $G = \mathrm{GL}_n\mathbb{C}$ and B is the subgroup of upper triangular matrices.
- The torus

$$T = \mathbb{T}^{n-1} = \{g = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \in \mathrm{GL}_n\mathbb{C} | \det g = 1\}$$

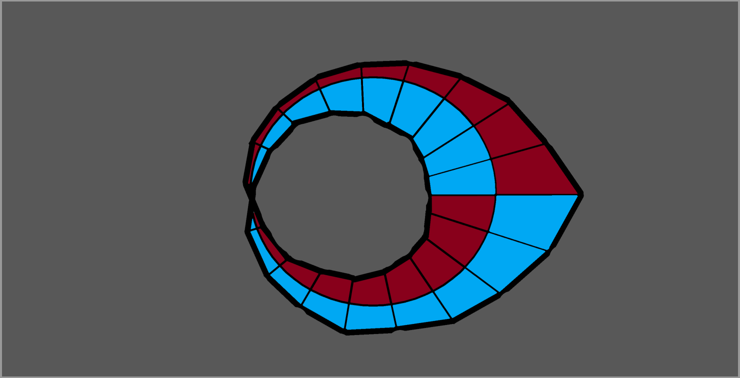
acts on $\mathrm{Fl}(\mathrm{n}, \mathbb{C})$ by left multiplication on matrices. The fixed points of this action are represented by the permutation matrices $(e_{\sigma(1)}\ e_{\sigma(2)}\ e_{\sigma(3)})_{\sigma \in \mathfrak{S}_n}$

- Example: the circle $T = S^1$ acts on $\mathrm{Fl}(2, \mathbb{C}) = \mathbb{C}\mathrm{P}^1 = S^2$ by rotation about a fixed axis (meaning 2 fixed points).

Line Bundles

- A topological **line bundle** over $\mathrm{Fl}(\mathrm{n}, \mathbb{C})$ is a continuous map $p : E \to \mathrm{Fl}(\mathrm{n}, \mathbb{C})$ such that for each flag x the preimage $p^{-1}(x)$ is a copy of $\mathbb{C}$ as a topological vector space, and x has an open neighborhood X such that the restricted map $p : p^{-1}(X) \to X$ is essentially the product projection $X \times \mathbb{C} \to X$.
- An equivariant line bundle has an action of T on both the total space and the base, that commute with the action.

Example



Representations

- Let's restrict our attention to a point $*$ of $\mathrm{Fl}(n, \mathbb{C})$. What does an equivariant bundle over $*$ look like? It consists of some vector space $\mathbb{C}^k$ along with a homomorphism $T \to \mathrm{GL}_k(\mathbb{C})$ called a representation of T.
- With the direct sum and tensor product operations, the set of representations of T *almost* has the structure of a ring, except that neither operation is strictly associative and there aren't additive inverses.
- If we take a representation up to some notion of equivalence compatible with the binary operations that includes isomorphisms of representation, we obtain the representation ring $R(T) \cong \mathbb{Z}[t_1^{\pm}, t_2^{\pm}, \ldots t_{n-1}^{\pm}]$

Equivariant K-Theory

- We can go through a similar process with equivariant vector bundles, with the correct notion of equivalence being that of *stable isomorphism*.
- Thus, to $\mathrm{Fl}(\mathrm{n}, \mathbb{C})$ we can associate a ring $K_T(\mathrm{Fl}(\mathrm{n}, \mathbb{C}))$ that models the algebra of vector bundles. Our goal is to understand the structure of this ring.

Main Takeaways

- $K_T(pt) \cong R(T)$
- $K_T(*) \cong \mathbb{Z}[t_1^{\pm}, \ldots, t_{n-1}^{\pm}]$.
- The inclusion of fixed points $\mathfrak{S}_n \hookrightarrow \mathrm{Fl}(\mathsf{n}, \mathbb{C})$ induces a map $K_T(\mathrm{Fl}(\mathsf{n}, \mathbb{C})) \to \bigoplus_{\sigma \in \mathfrak{S}_n} R(T) \cong \bigoplus_{\mathfrak{S}_n} \mathbb{Z}[t_1^{\pm}, \ldots, t_{n-1}^{\pm}]$.

Restriction to Fixed Points

Theorem

The function $K_T(\mathrm{Fl}(n,\mathbb{C})) \to \bigoplus_{\mathfrak{S}_n} \mathbb{Z}[t_1^{\pm}, \ldots, t_{n-1}^{\pm}]$ induced by the inclusion of the fixed points into $\mathrm{Fl}(n,\mathbb{C})$ is an injective ring map.

- Thus, we can think of a vector bundle $X \in K_T(\mathrm{Fl}(\mathrm{n},\mathbb{C}))$ as determined by its restriction to the fixed points of the T-action, which is just a collection of representations of T.

Tensor Product Structure

Theorem

(Steinberg, Kostant-Kumar)

$$K_T(Fl(n;\mathbb{C})) \cong R(T) \otimes_{R(T)^{\mathfrak{S}_n}} R(T)$$

where $R(T)^{\mathfrak{S}_n}$ *denotes the subring fixed by the action of* $\mathfrak{S}_n$ *induced by its action on* $\{t_1, t_2, \ldots, t_n\}$ *where*

$$t_n = (t_1 t_2 \ldots t_{n-1})^{-1}$$

- This gives us a concise way of writing elements of the K-theory, and is suggestive of how to find a basis for the ring as a module over $R(T)$.

$\mathrm{Fl}(3, \mathbb{C})$

- There are two bases for the equivariant K-theory $K_T(\mathrm{Fl}(3, \mathbb{C}))$ that we considered.
- The Line Bundle basis, which relies on the interpretation of $K_T(\mathrm{Fl}(3, \mathbb{C}))$ as the module $R(T) \otimes_{R(T)^{\mathfrak{S}_n}} R(T)$ as well as something called a divided difference operator.
- The Schubert basis is determined by a certain class of subvarieties of $\mathrm{Fl}(3, \mathbb{C})$. These are the Schubert varieties, and they are also indexed by $\mathfrak{S}_n$.

Goal #1

We discussed two relevant bases of $K_T(\mathrm{Fl}(\mathrm{n}, \mathbb{C}))$: Schubert basis, and a basis of line bundles.

- Write the Schubert classes as linear combinations of (equivariant) line bundles over $\mathrm{Fl}(3, \mathbb{C})$.
- The coefficients are Laurent polynomials.

Two bases of $K_T(\mathrm{Fl}(3, \mathbb{C}))$:

$$\{\text{Schubert classes}\} \leftrightarrow \{\text{Line bundles}\}$$

To be specific: if $\mathcal{O}_{X_w}$ is a Schubert class, then

$$\mathcal{O}_{X_w} = \sum_i \beta_i L_i$$

where $L_1, \ldots, L_6$ are the 6 line bundles in the line bundle basis, and $\beta_1, \ldots, \beta_6$ are Laurent polynomials in t_1, t_2, t_3.

Recall that a K-theory class $[\alpha]$ is also a collection of $3! = 6$ Laurent polynomials, so $[\alpha] = (\alpha_\sigma)_{\sigma \in \mathfrak{S}_n}$

$$X_{[213]} = [t_1 t_2 t_3^{-2} - t_1 t_3^{-1} - t_2 t_3^{-1} + 1, t_1 t_2 t_3^{-2} - t_1 t_3^{-1} - t_2 t_3^{-1} + 1, 0, 0, 0, 0]$$

$$1 \otimes 1 = [1, 1, 1, 1, 1, 1]$$
$$1 \otimes (t_1 + t_2) = [t_1 + t_2, t_2 + t_1, t_1 + t_3, t_2 + t_3, t_3 + t_1, t_3 + t_2]$$
$$1 \otimes t_1 t_2 = [t_1 t_2, t_2 t_1, t_1 t_3, t_2 t_3, t_3 t_1, t_3 t_2]$$

Here is an example of how we can write the Schubert Basis in terms of the Line Bundle basis.

$$X_{[213]} = 1 \otimes 1 \; - \; t_1 t_2 (1 \otimes (t_1 + t_2)) \; + \; t_1^2 t_2^2 (1 \otimes t_1 t_2)$$

where $t_3 = (t_1 t_2)^{-1}$.

Code and Challenges

```
def fixed_point_restr(g,f):
        Ps = [[1,2,3],[2,1,3],[1,3,2],[2,3,1],[3,1,2],[3,2,1]]
        Ps_in_cyclic = [Permutation(p).to_cycles() for p in Ps]
        FixedP = [KK(g)*permute_polynomial(p, f) for p in Ps_in_cyclic]
        return FixedP;

def line_bundle_matrix():
        starting_polynomials = [perm_delt(p) for p in Permutations(3)]
        starting_polynomials.reverse()
        list_of_gons = [fixed_point_restr(1,f) for f in starting_polynomials]
        list_of_gons = simplify(list_of_gons)
        return Matrix(KK, list_of_gons).transpose();
```

A snippet of the code

To handle writing the Schubert Basis in terms of the Line Bundle basis, we used SageMath. We wrote a program to generate our bases. We then represented our bases as matrices of Laurent Polynomials and used Sage's linear algebra tools find a change of basis matrix.

The Change of Basis Matrix

[321]	[312]	[231]	[132]	[213]	[123]
1	1	1	1	1	1
0	0	$-t_1 t_2$	$-t_1 t_2 - t_1 t_3$	0	$-t_1 t_3$
0	0	0	0	$-t_1 t_2$	$-t_1 t_2$
0	$-t_1$	0	0	$t_1^2 t_2^2$	$t_1^2 t_2^2 + t_1$
0	0	0	t_1	0	t_1
0	0	0	0	0	$-t_1^2 t_2$

Change of Basis

$$X_{[123]} = 1\otimes 1 \; - \; t_2\otimes t_1 \; - \; t_1 t_2\otimes(t_1 + t_2) \; + \; (t_1^2 t_2^2 + t_1)\otimes t_1 t_2$$
$$+ \; t_1\otimes t_1^2 \; - \; t_1^2 t_2\otimes t_1^2 t_2$$
$$X_{[213]} = 1\otimes 1 \; - \; t_1 t_2\otimes(t_1 + t_2) \; + \; t_1^2 t_2^2\otimes t_1 t_2$$
$$X_{[132]} = 1\otimes 1 \; - \; (t_1 t_2 + t_1 t_3)\otimes t_1 \; + \; t_1\otimes t_1^2$$
$$X_{[231]} = 1\otimes 1 \; - \; t_1 t_2\otimes t_1$$
$$X_{[312]} = 1\otimes 1 \; - \; t_1\otimes t_1 t_2$$
$$X_{[321]} = 1\otimes 1$$