Dynamics of a group action on Character Varieties

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Overview

- We are interested in the varieties given by the zero sets of a certain polynomial κ over the affine 3-space $\mathbb{F}_q^3$, where $\mathbb{F}_q$ is a finite field. In particular, we look at the dynamics of the action of a subgroup of the automorphism group of this variety, denoted by Γ, on the κ-varieties.
- The goal of the project is to examine the conjecture that this action is arithmetically ergodic in the sense that, as $q \to \infty$, the action becomes 'almost transitive'.
- We are trying to use the approach of dividing the variety into conics which was used by Bourgain, Gamburd and Sarnak to address a similar problem for a specific variety using number theoretic methods.

Definitions

- Let $\mathbb{F}_q^3$ be the affine space for a finite field of order $q = p^n$, with p a prime.
- The varieties we consider are the zeros of the polynomial $\kappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2 - \lambda$ in $\mathbb{F}_q^3$ for some $\lambda \in \mathbb{F}_q$.
- Let $\Gamma = \langle \eta, \tau, \iota \rangle$ be a group of automorphisms that preserve the zero set of κ where η, τ and ι are given by:
  - $\eta(x, y, z) = (z, y, zy - x)$
  - $\iota(x, y, z) = (x, y, xy - z)$
  - $\tau(x, y, z) = (y, x, z)$

Then Γ acts on the variety.

Definitions

We say the orbit of a point in our variety is the collection of those points in $\mathbb{F}_q^3$ which are accessible by applying a morphism in Γ to that first point. For $v = (x, y, z) \in \mathbb{F}_q^3$, define the orbit of v as

$$\mathrm{Orb}_\Gamma(v) = \{w \in \mathbb{F}_q^3 \ : w = \alpha(v) \text{ for some } \alpha \in \Gamma\}.$$

Visualizations

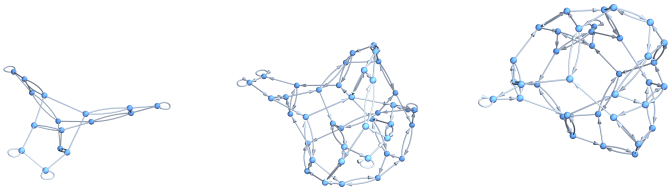


Figure: Induced actions on some varieties in $\mathbb{F}_3^3$, $\mathbb{F}_5^3$, and $\mathbb{F}_7^3$

Project Goal

- Ultimately, we want to show this action is "Arithemetically Ergodic"
- Roughly speaking, as the order of the field grows, we should have an effectively transitive induced action
- That is, for almost any point in the variety, we can get to almost any other via an element of Γ

Degrees of Success with Transitivity

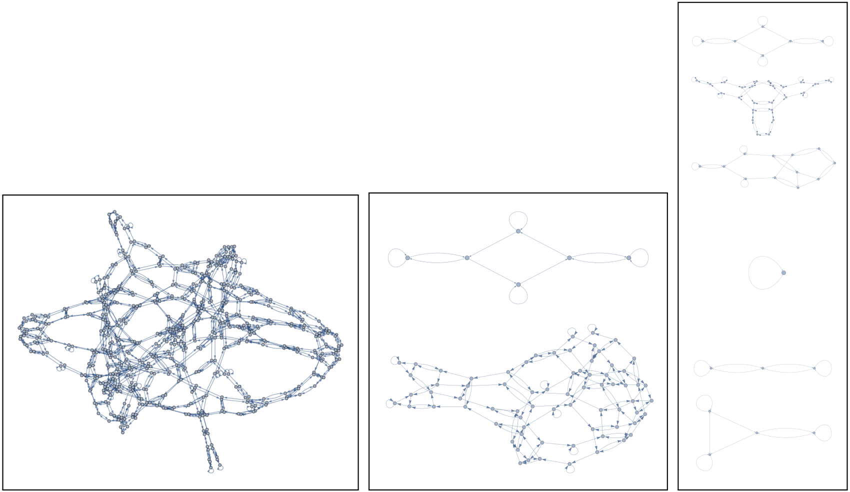


Figure: The Good, the Bad, and the Ugly

Approach with Conics

- Bourgain, Gamburd and Sarnak showed that the action of Γ is transitive on the affine surface $\mathbb{X}(\mathbb{Z}/p\mathbb{Z})$ in $(\mathbb{Z}/p\mathbb{Z})^3$ given by $x_1^2 + x_2^2 + x_3^2 - 3x_1x_2x_3 = 0$ for most primes p.
- Let $X^*(p) = \mathbb{X}(\mathbb{Z}/p\mathbb{Z}) \setminus \{(0,0,0)\}$.
- For $i \in \{1,2,3\}$, define conic sections of the variety by

$$C_i(a) = \{(x_1, x_2, x_3) \in (\mathbb{Z}/p\mathbb{Z})^3 : x_i = a\} \cap X^*(p).$$

Conics

- Define the incidence graph $I(p)$ as the graph whose vertices are the above defined cones such that the number of edges between $C_j(a)$ and $C_j(b)$ is $|C_j(a) \cap C_j(a)|$.
- Then $I(p)$ is connected for all large enough p, and $diam(I(p)) = 2$.
- Define $rot(3x_1)\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ 3x_1x_3 - x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 3x_1 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$.

Theorem

If $x = (x_1, x_2, x_3)$ is in $X^*(p)$ and for some $j \in \{1, 2, 3\}$ the order of the induced rotation $rot(3x_j)$ is at least $p^{\frac{1}{2}+\delta}(\delta > 0$ fixed), then x is joined to a point y in $X^*(p)$ one of whose induced rotations is of maximal order.

Cage

- A point $x = (x_1, x_2, x_3) \in X^*(p)$ is called *maximal* if the order of $rot(3x_j)$ is maximal for some j.
- The cage is the set of maximal elements in $X^*(p)$.
- The cage is connected.
- We define $C(p)$ to be the connected component of $X^*(p)$ under the Γ action that contains the cage and hence is the largest component.

Theorem (Bourgain, Gamburd and Sarnak)

Fix $\epsilon > 0$. Then for p sufficiently large

$$|X^*(p) \setminus C(p)| \leq p\epsilon$$

(note that $|X^*(p)| \sim p^2$), and any Γ orbit $\mathcal{D}(p)$ satisfies

$$|\mathcal{D}(p)| \gg (logp)^{\frac{1}{3}}.$$

Exceptional Orbits

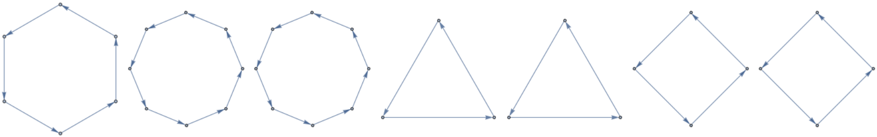
- An exceptional orbit is one whose growth is bounded as the order of the field increases.
- In an earlier semester, Marvin Castellon identified six reoccuring exceptional orbits.
- Marvin conjectured that these six exceptional orbits comprise of all possible orbits which can appear in the variety over $\mathbb{F}_p^3$ except for one nearly transitive orbit. This “exceptional orbit conjecture” would imply the arithmetic ergodicity conjecture.

Some Observations

- The map $f(x, y, z) = (3z, 3y, 3z)$ gives an isomorphism of varieties between the Markoff surface and the κ-variety for $\lambda = -2$ and $p \neq 3$.
- "η-like" elements preserve the conics.
- We know the possible sizes of conics as well as exactly when they will occur.

Some Observations

- For $p = 7$, the η orbits are as follows:



- Among these, each of the hexagon and the octagons are conics and the union of the triangles as well as the union of the squares give the other two conics. This shows that η need not act transitively on the conics.
- There are connections between the conjectured existence of certain exceptional orbits and the appearance of certain types of conics. Some orbits can appear only if a certain type of conic appears (presuming the exceptional orbits conjecture is true).

Future Directions

- Verify whether the approach of conics works for different values of λ.
- Try to approach transitivity by looking for elements of Γ that allows transition between η orbits of a single orbit as well as between different conics.
- Determine whether the Exceptional Orbits Conjecture (or a weaker variant of it) is true.

Acknowledgements

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