Change of Basis in the Equivariant K-theory of Flag Manifolds

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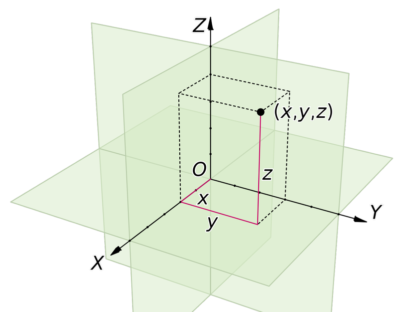
Introduction to Flags

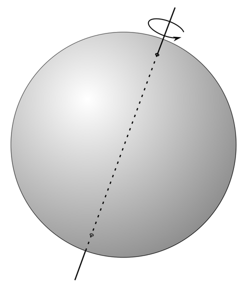


A (complete) **flag** on $\mathbb{C}^n$, is a maximal chain of vector subspaces of $\mathbb{C}^n$, i.e. a chain $\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n$ with each V_i having dimension i. The set of all flags of $\mathbb{C}^n$ is denoted by $Fl(n;\mathbb{C})$, and is called a flag variety. Another characterization of $Fl(n;\mathbb{C})$ is as a quotient of the group of invertible $n \times n$ matrices. Using the standard basis of $\mathbb{C}^n$, $Fl(n;\mathbb{C}) = GL_n(\mathbb{C})/B$ where $B \subset GL_n(\mathbb{C})$ is the subgroup of upper triangular matrices.

A Group Action on Flag Manifolds

Let $T^n \simeq S^1 \times S^1 \times \cdots \times S^1$ be the **torus** of dimension n.

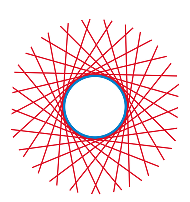
- T^{n-1} can also be seen as the set of diagonal $n \times n$ matrices with unit modulus entries in the diagonal and determinant 1.
- As $T := T^{n-1} \subset Gl_n(\mathbb{C})$ acts on $G = GL_n(\mathbb{C})$ (by matrix multiplication), so does it act on $Fl(n,\mathbb{C})$ when considering the flag manifold as the quotient G/B.

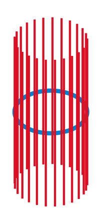


A sphere rotating about an axis is an example of a group action

Line Bundles

A *line bundle* over $Fl(n;\mathbb{C})$ is a continuous map $p : E \to Fl(n,\mathbb{C})$ such that over each flag $x = (\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n)$ the fiber $p^{-1}(x)$ is a copy of $\mathbb{C}$ as a topological vector space, and moreover it is possible to find an open cover of $Fl(n;\mathbb{C})$ over which the restrictions of p to the elements of the cover are essentially a product projection.





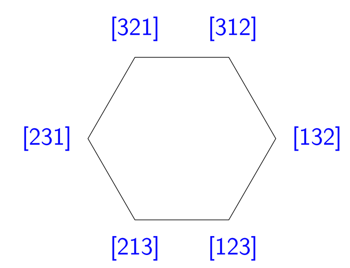
As we have a group action on $Fl(n;\mathbb{C})$, we want to look at ways to "lift" the action to a line bundle in way that is compatible with the bundle structure. A line bundle with a prescribed lift of a group action is called an *equivariant line bundle*. For each equivariant line bundle E, let $[E]$ denote the set of bundles that are stably isomorphic to E. The set of formal differences $K_T(Fl(n;\mathbb{C})) = \{[E]-[F] | E, F \text{are equivariant bundles}\}$ is given the structure of a ring by direct sum and tensor product of bundles, and is called the equivariant K-theory ring of $Fl(n;\mathbb{C})$.

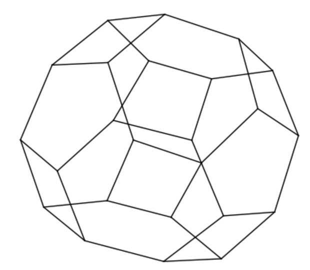
Fixed Points

- A natural question to ask is whether there are any flags that are fixed by all the matrices in T? As it turns out, this is very useful for computing the equivariant K-theory of $Fl(n;\mathbb{C})$

Remark. The fixed points of the T-action can be represented by the matrices in the defining representation of $\mathfrak{S}_n$, the symmetric group on n letters.

T^2 acts on $Fl(3;\mathbb{C})$; on the left is a representation of the fixed point set with the weak Bruhat order on $\mathfrak{S}_3$, and on the right is the fixed point set if we go up a dimension.





Computational Tools

Theorem
The natural map $K_T(Fl(n;\mathbb{C})) \to \bigoplus_{\mathfrak{S}_n} K_T(pt)$ induced by the inclusion of the T-fixed points into the flag manifold is an injective ring map. A vector bundle is taken to its restriction on the set of fixed points.

Theorem
(Steinberg, Kostant-Kumar)

$$K_T(Fl(n;\mathbb{C})) \cong R(T) \otimes_{R(T)^{\mathfrak{S}_n}} R(T)$$

where $R(T) = K_T(pt)$ is the representation ring of T, and $R(T)^{\mathfrak{S}_n}$ denotes the subring fixed by the obvious action of $\mathfrak{S}_n$ on the polynomial ring

$$R(T^n) \cong \mathbb{Z}[t_1, \dots t_n, t_1^{-1}, \dots t_n^{-1}]$$

the ring of Laurent polynomials on $t_1, \dots t_n$ over $\mathbb{Z}$.

Line Bundle and Schubert Bases

Below are tables that represent both the Line Bundle and Schubert bases on $Fl(3;\mathbb{C})$. Each row of each table represents an element of each basis when restricted to the fixed points.

	[123]	[213]	[132]	[231]	[312]	[321]
$1 \otimes 1$	1	1	1	1	1	1
$1 \otimes t_1$	t_1	t_2	t_1	t_2	t_3	t_3
$1 \otimes (t_1 + t_2)$	$t_1 + t_2$	$t_1 + t_2$	$t_1 + t_3$	$t_2 + t_3$	$t_1 + t_3$	$t_2 + t_3$
$1 \otimes t_1 t_2$	$t_1 t_2$	$t_1 t_2$	$t_1 t_3$	$t_2 t_3$	$t_1 t_3$	$t_2 t_3$
$1 \otimes t_1^2$	t_1^2	t_2^2	t_1^2	t_2^2	t_3^2	t_3^2
$1 \otimes t_1^2 t_2$	$t_1^2 t_2$	$t_1 t_2^2$	$t_1^2 t_3$	$t_2^2 t_3$	$t_1 t_3^2$	$t_2 t_3^2$

Line Bundle Basis

[123]	[213]	[132]	[231]	[312]	[321]	
$*$	0	0	0	0	0	$X_{[123]}$
$t_1 t_2 t_3^{-2} - t_1 t_3^{-1} - t_2 t_3^{-1} + 1$	$t_1 t_2 t_3^{-2} - t_1 t_3^{-1} - t_2 t_3^{-1} + 1$	0	0	0	0	$X_{[213]}$
$t_1^2 t_2^{-1} t_3^{-1} - t_1 t_3^{-1} - t_1 t_2^{-1} + 1$	0	$t_1^2 t_2^{-1} t_3^{-1} - t_1 t_3^{-1} - t_1 t_2^{-1} + 1$	0	0	0	$X_{[132]}$
$-t_1 t_3^{-1} + 1$	$-t_2 t_3^{-1} + 1$	$-t_1 t_3^{-1} + 1$	$-t_2 t_3^{-1} + 1$	0	0	$X_{[231]}$
$-t_1 t_3^{-1} + 1$	$-t_1 t_3^{-1} + 1$	$-t_1 t_2^{-1} + 1$	0	$-t_1 t_2^{-1} + 1$	0	$X_{[312]}$
1	1	1	1	1	1	$X_{[321]}$

Schubert Basis

$$* = -t_1^2 t_3^{-2} + t_1 t_2 t_3^{-2} + t_1^2 t_2^{-1} t_3^{-1} - t_2 t_3^{-1} - t_1 t_2^{-1} + 1$$

The Schubert Basis

While the symmetric group $\mathfrak{S}_n$ indexes the fixed points of the T action on $Fl(n;\mathbb{C})$, it also indexes a special set of subvarieties called Schubert varieties. This set also gives a basis for $K_T(Fl(n;\mathbb{C}))$, which is the Schubert basis in the above table.

While we tailored our code for the Schubert varieties of $Fl(3;\mathbb{C})$, we have a formula for forming the fixed point restriction of something called the opposite Schubert varieties in any dimension.

Change of Basis

```
def fixed_point_restr(g,f):
        Ps = [[1,2,3],[2,1,3],[1,3,2],[2,3,1],[3,1,2],[3,2,1]]
        Ps_in_cyclic = [Permutation(p).to_cycles() for p in Ps]
        FixedP = [KK(g)*permute_polynomial(p, f) for p in Ps_in_cyclic]
        return FixedP;

def line_bundle_matrix():
        starting_polynomials = [perm_delt(p) for p in Permutations(3)]
        starting_polynomials.reverse()
        list_of_gons = [fixed_point_restr(1,f) for f in starting_polynomials]
        list_of_gons = simplify(list_of_gons)
        return Matrix(KK, list_of_gons).transpose();
```

A snippet of the code

The code creates a matrix of coefficients that describes the decomposition of each Schubert basis element in terms of the Line Bundles basis. The result is

$X_{[123]} = 1 \otimes 1 - t_2 \otimes t_1 - t_1 t_2 \otimes (t_1 + t_2) + (t_1^2 t_2^2 + t_1) \otimes t_1 t_2 + t_1 \otimes t_1^2 - t_1^2 t_2 \otimes t_1^2 t_2$

$X_{[213]} = 1 \otimes 1 - t_1 t_2 \otimes (t_1 + t_2) + t_1^2 t_2^2 \otimes t_1 t_2$

$X_{[132]} = 1 \otimes 1 - (t_1 t_2 + t_1 t_3) \otimes t_1 - t_1 \otimes t_1^2$

$X_{[231]} = 1 \otimes 1 - t_1 t_2 \otimes t_1$

$X_{[312]} = 1 \otimes 1 - t_1 \otimes t_1 t_2$

$X_{[321]} = 1 \otimes 1$

References

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2. R. Steinberg, On a theorem of Pittie, Topology 14 (1975), 173177.
3. Graham, William; Kreiman, Victor Excited Young diagrams, equivariant K-theory, and Schubert varieties. Trans. Amer. Math. Soc. 367 (2015), no. 9, 65976645.