Dynamics of group action on Character Varieties

Julian Benali, Seth Lee, Cigole Thomas
Faculty advisor : Sean Lawton

Mason Experimental Geometry Lab

December 10, 2018

Abstract

- We are interested in the varieties of the affine 3 -space $\mathbb{F}_q^3$, where $\mathbb{F}_q$ is a finite field, given by the zero sets of a certain polynomial κ. In particular, we look at the dynamics of the action of a subgroup of the automorphism group of κ denoted by Γ, on the κ-varieties.
- The goal of the project is to examine the conjecture that this action is arithmetically ergodic in the sense that, as $q \to \infty$, the action becomes 'almost transitive'.
- We are trying to use the approach of dividing the variety into conics which was used by Bourgain, Gamburd and Sarnak to address a similar problem for a specific variety using number theoretic methods.

Introduction

- Let $\mathbb{F}_q^3$ be the affine space for a finite field of order $q = p^n$, with p a prime.
- The varieties we consider are the zeros of the polynomial $\kappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2 - \lambda$ in $\mathbb{F}_q^3$ for some $\lambda \in \mathbb{F}_q$.
- Let $\Gamma = < \eta, \tau, \iota >$ be the group of polynomial automorphisms that preserve κ where η, τ and ι are given by:
  - $\eta(x, y, z) = (z, y, zy - x)$
  - $\iota(x, y, z) = (x, y, xy - z)$
  - $\tau(x, y, z) = (y, x, z)$
- Then Γ acts on the variety.
- For $v = (x, y, z) \in \mathbb{F}_q^3$, define the orbit of v as

$$\mathrm{Orb}_\Gamma(v) = \{w \in \mathbb{F}_q^3 : w = \alpha(v) \text{ for some } \alpha \in \Gamma\}.$$

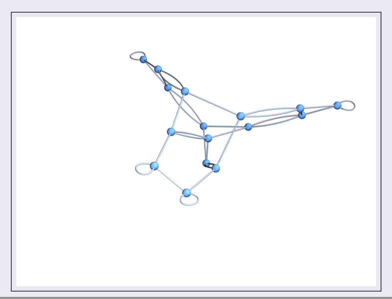
The Ergodicity Conjecture

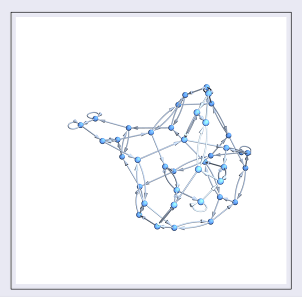
- We want to show that as $q \to \infty$, almost all elements in the varieties (except for a set of "measure zero") form a single orbit .

Orbits Visualized

We used Mathematica to visualize the orbits of the above defined action. Some of the orbits are shown below.

Orbits of induced actions on some varieties in $\mathbb{F}_3^3$ and $\mathbb{F}_5^3$.





Degrees of Sucess with Transitivity

On examining the orbits for smaller values of prime p, we observe varying degrees of transitivity as measured by the number and size of the orbits. A few typical examples are demonstrated below.

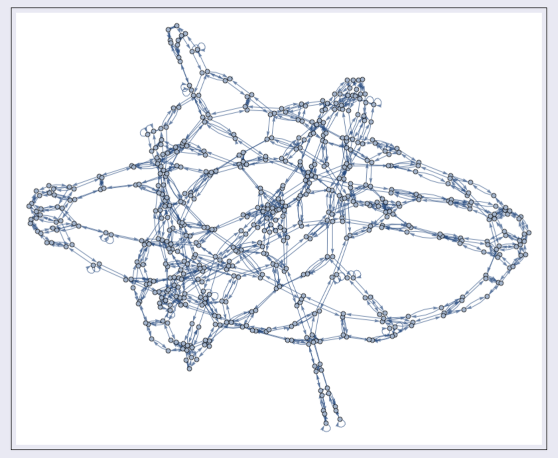


Figure: The Good

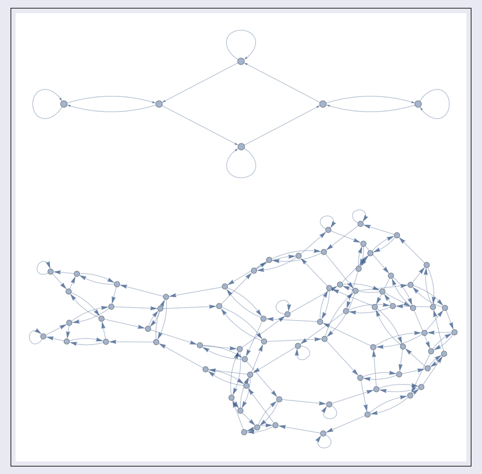


Figure: The Bad

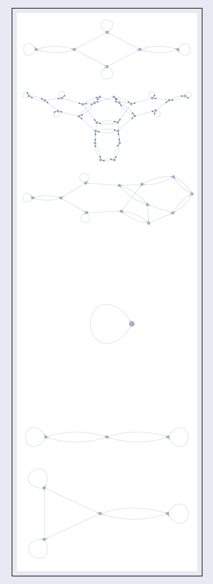


Figure: The Ugly

Approach with Conics

- Bourgain, Gamburd and Sarnak showed that the action of Γ is transitive on the affine surface $\mathbb{X}(\mathbb{Z}/p\mathbb{Z})$ in $(\mathbb{Z}/p\mathbb{Z})^3$ given by $x_1^2 + x_2^2 + x_3^2 - 3x_1x_2x_3 = 0$ for most primes p.
- Let $X^*(p) = \mathbb{X}(\mathbb{Z}/p\mathbb{Z}) \setminus \{(0, 0, 0)\}$.
- Define conic sections of the variety

$$C_i(a) = \{(x_1, x_2, x_3) \in (\mathbb{Z}/p\mathbb{Z})^3 : x_i = a\} \cap X^*(p).$$

- Define the incidence graph $I(p)$ as the graph with the above defined cones as vertices and number of edges between $C_j(a)$ and $C_j(b) = |C_j(a) \cap C_j(a)|$.
- Then $I(p)$ is connected and $diam(I(p)) = 2$.
- Define $rot(3x_1)\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ 3x_1x_3 - x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 3x_1 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$.

Theorem

If $x = (x_1, x_2, x_3)$ is in $X^*(p)$ and for some $j \in \{1, 2, 3\}$ the order of the induced rotation $rot(x_j)$ is at least $p^{\frac{1}{2}+\delta}(\delta > 0$ fixed), then x is joined to a point y in $X^*(p)$ one of whose induced rotations is of maximal order.

- A point $x = (x_1, x_2, x_3) \in X^*(p)$ is called *maximal* if $ord(rot(x_j))$ is maximal for some j.
- A cage is a set of maximal elements in $X^*(p)$.
- The cage is connected.
- We define $C(p)$ to be the connected component of $X^*(p)$ under the Γ action that contains the cage and hence is the largest component.

Theorem (Bourgain, Gamburd and Sarnak)

Fix $\epsilon > 0$. Then for p large there is a Γ orbit $C(p)$ in $X^*(p)$ for which

$$|X^*(p) \setminus C(p)| \leq p\epsilon$$

(note that $|X^*(p)| \sim p^2$), and any Γ orbit $\mathcal{D}(p)$ satisfies

$$|\mathcal{D}(p)| \gg (logp)^{\frac{1}{3}}$$

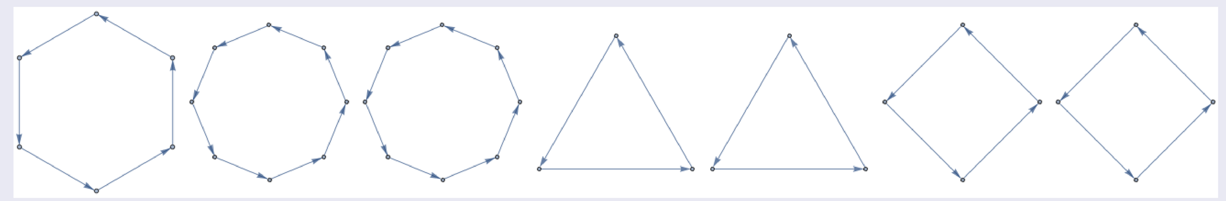
.

Exceptional Orbits

- An exceptional orbit is one whose growth is bounded as the order of the field increases.
- In an earlier semester, Marvin Castellon identified six reoccuring exceptional orbits.
- Marvin conjectured that these six exceptional orbits comprise of all possible orbits which can appear in the variety for $\mathbb{F}_p$ except for one nearly transitive orbit. This "exceptional orbit conjecture" would imply the arithmetic ergodicity conjecture.

Some observations

- The map $f(x, y, z) = (3z, 3y, 3z)$ gives an isomorphism of varieties between the Markoff surface and the variety defined by $\lambda = -2$ and $p \neq 3$.
- "η-like" elements preserve the conics.
- We know the possible sizes of conics as well as exactly when they will occur.
- For $p = 7$, the η orbits are as follows:



Among these, each of the hexagon and the octagons are conics and the union of the triangles as well as the union of the squares give the other two conics. This shows that η need not act transitively on the conics.
- There are connections between the conjectured existence of certain exceptional orbits and the appearance of certain types of conics. Some orbits can appear only if a certain type of conic appears (presuming the exceptional orbits conjecture is true).

Future Work

- Verify whether the approach of conics works for different values of λ.
- Try to approach transitivity by looking for elements of Γ that allows transition between η orbits of a single orbit as well as between different conics.
- Determine whether the Exceptional Orbits Conjecture (or a weaker variant of it) is true.

Acknowledgements

We would like to thank Dr. Sean Lawton for the direction and support he has given us in this project as well as all the previous participants of this project for their contributions.