Asymptotic Dynamics on Arithmetic Curves

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May 4, 2018

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Optional Background

- Imagine a once-punctured torus (call it X)
- Now imagine all the ways you can manipulate it continuously without it self-intersecting (These are homotopy classes in $\mathrm{Homeo}(X, X)$)
- Some transformations can not be obtained from another by this kind of deformation (in a sense they are not connected)
- This is called the mapping class group, and for us it is isomorphic to $\mathrm{Out}(F_2)$
- We are looking at the action of this group on a seemingly simpler object called a character variety. In our case it is affine 3-space over a finite field, or rather a cube of points with oddly specific side lengths

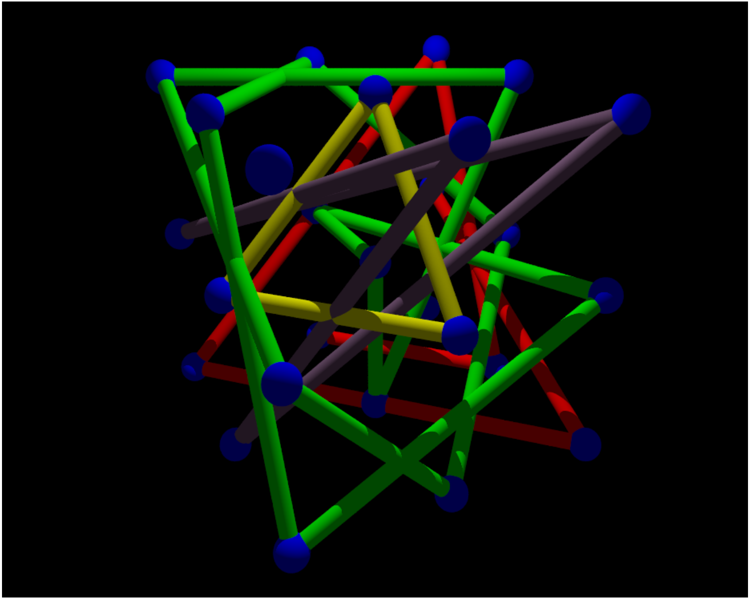
Definitions

- We denote the affine 3-space, $\mathbb{F}_q^3$.
- It turns out $\mathrm{Out}(F_2)$ is generated by the three elements we call η, τ, and ι.
- We define these elements as follows:

$$\eta \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ yz - x \end{pmatrix} \qquad \tau \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ z \end{pmatrix} \qquad \iota \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ xy - z \end{pmatrix}$$

- All the finite combinations of these actions and their inverses form a group we call Γ.
- We define the orbit of Γ on an element in $\mathbb{F}_q^3$ as the result of taking all of the elements of Γ and applying it to that point.
- One of the breakthroughs this semester was discovering $\iota = \tau\eta\tau\eta^{-1}\tau\eta$ as elements of the induced action of Γ over the character variety, but *not* as elements of $\mathrm{Out}(F_2)$.

$\eta\tau$ in $\mathbb{F}_3^3$



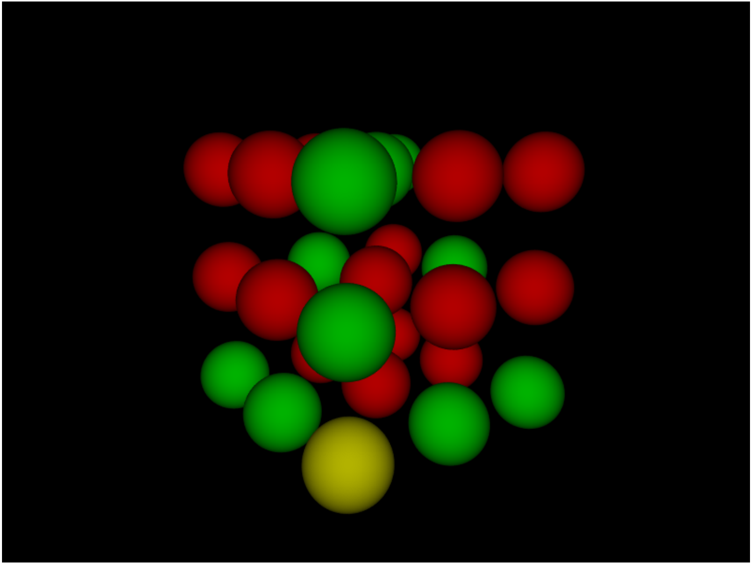
Definitions (cont.)

- The action of Γ preserves the polynomial κ, defined as $\kappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2$.
- We define a κ variety for some $\lambda \in \mathbb{F}_q$ as $\kappa^{-1}(\lambda) = \{(x, y, z) \in \mathbb{F}_q^3 : \kappa(x, y, z) = \lambda\}$.

In order to measure the size of these orbits, we define the $\mathcal{L}$ functions:

- $\mathcal{L}(\lambda, \mathbb{F}_q)$ denotes the length of the largest orbit in a κ variety, $\kappa^{-1}(\lambda)$,
- $\mathcal{L}^{max}(\mathbb{F}_q)$ denotes the length of the largest orbit in $\mathbb{F}^3$,
- $\mathcal{L}^{avg}(\mathbb{F}_q)$ denotes the average of the $\mathcal{L}$ function values. This makes sense since there are only finitely many κ varieties.

Character Variety for $p = 3$



Arithmetic Ergodicity

Ergodicity in dynamics is a description of how large orbits get.

- We know that the level sets of the κ varieties are quadratic in q.
- We say that the action of Γ is arithmetically ergodic if

$$\lim_{q \to \infty} \frac{\mathcal{L}^{max}(\mathbb{F}_q)}{q^2} = 1.$$

- We want to determine whether the action of Γ is arithmetically ergodic.

Progress on AE

Most of our progress is based on verifying arithmetic ergodicity computationally.

- We wrote code to compute these orbits over prime fields and general finite fields (we specialize with prime fields though) .
- Our data strongly supports the conclusion of arithmetic ergodicity .
- We have shown the existence of orbits of at least linear growth (η gives us such orbits).
- The chaotic nature of this action makes it hard to analyze the size of the orbits.

Instead of just looking at the size of the largest orbits, we can look at how many orbits there are per κ variety.

The Exceptional Orbit Conjecture

It appears to be true that **ALL** orbits not on $\kappa^{-1}(2)$ fall into one of the following types:

1. $\mathrm{Orb}_\Gamma(0,0,0)$ of order 1 ($\lambda = -2$),
2. $\mathrm{Orb}_\Gamma(a,0,0)$ of order 6 if $a \neq 0$ when $\left(\frac{\lambda+2}{p}\right) = 1$,
3. $\mathrm{Orb}_\Gamma(-1,0,1)$ of order 16 when $\lambda = 0$,
4. $\mathrm{Orb}_\Gamma(0,1,\sqrt{2})$ of order 36 when $p \equiv \pm 1 (\mathrm{mod}\ 8)$ and $\lambda = 1$,
5. $\mathrm{Orb}_\Gamma(0,1,\frac{1+\sqrt{5}}{2})$ of order 40 when $p \equiv \pm 1 (\mathrm{mod}\ 10)$ and $\lambda = \frac{1\pm\sqrt{5}}{2}$,
6. $\mathrm{Orb}_\Gamma(1,1,\frac{1+\sqrt{5}}{2})$ of order 72 when $p \equiv \pm 1 (\mathrm{mod}\ 10)$ and $\lambda = 1$.

We have reason to believe after removing these exceptions that the remaining elements form a single orbit.

EOC Significance

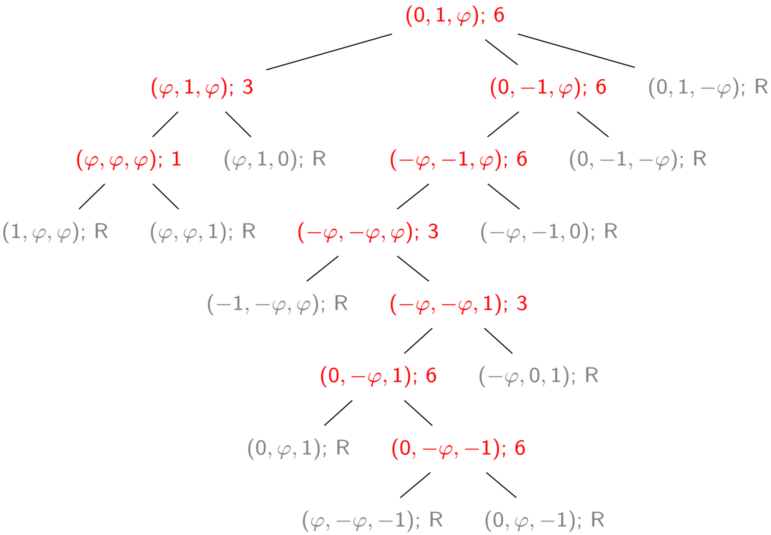
Exceptional Orbit Conjecture $\Rightarrow$ Arithmetic Ergodicity

- This gives us a counting function for the number of orbits in a κ variety that we can check numerically.

Why we believe

- We computed the orbit counting function, and then counted the orbits (computer). The result was positive for more than two weeks until the computer decided it needed to restart.
- Manual calculations show the six exceptional orbit types do truly close up to the sizes listed.
- If there were more exceptional orbits, we would expect them to have shown up by now.
- Looking at the $\mathbb{C}$ points under the Γ action, we see that the exceptional orbits are contained within the box of infinity norm two.
- More specifically, this is the SU(2) character variety.

An Orbit of Order 40



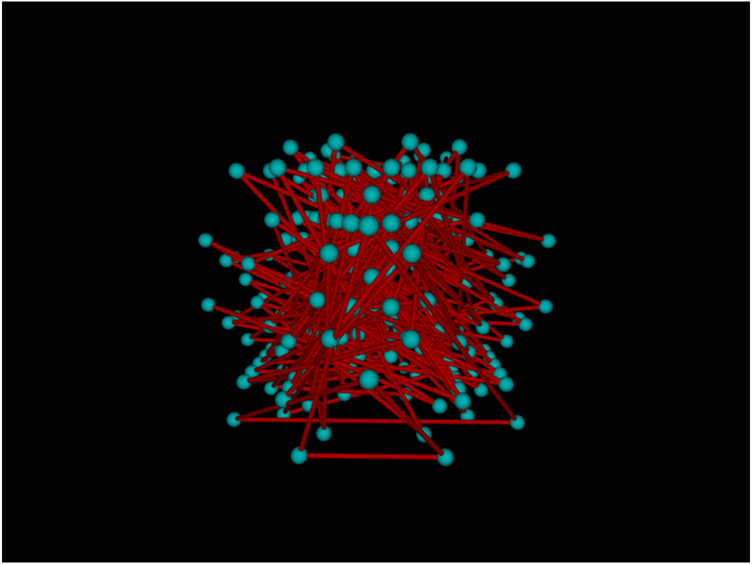
$q \log q$ Conjecture

- Data suggests that the growth rates of $\eta^k \tau$ orbits ($k \neq 0$) are asymptotic to $q \log q$.
- For those with some topology background, we believe that these actions correspond to Dehn Twists on the once punctured torus.

$\eta^k\tau$ Conjecture

- Data suggests that the union of $\eta^k\tau$ orbits is the full Γ orbit starting from any point.

$\eta\tau$ in $\mathbb{F}_{13}^3$



Progress on the $q \log q$ and $\eta^k \tau$ Conjectures

- Little progress has been made here other than computational data.
- Interestingly, we keep bumping into deep, open problems in number theory! (Pisano Periods, Reimann Hypothesis, etc.)

Future Work

- We plan to get our hands dirty manually computing orbits, and really get to know the underlying structure.
- Generalise the EOC to finite fields.
- Work on classifying exceptional orbits over $\kappa^{-1}(2)$

Acknowledgment

- We thank Dr. Lawton for his guidance and providing the opportunity to work at MEGL.
- We also thank the GMU Math Department, and GMU math community for an exciting place to collaborate and explore math!

Q&A

Questions?
Comments?