Orbits on the Finite Field Points of the $SL_2(\mathbb{C})$ Character Variety of F_2

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Abstract

- The automorphism group of the polynomial $\kappa(x,y,z) = x^2 + y^2 + z^2 - xyz - 2$ over a finite field $\mathbb{F}_p^3$ has a subgroup Γ, consisting of polynomial automorphisms, whose orbit lengths are of particular fascination. The group Γ is generated by the automorphisms ι, τ, η and acts on the variety $\kappa^{-1}\lambda$, for λ in $\mathbb{F}$. We are interested in the length of the longest orbit, denoted $\mathcal{L}_{\langle w \rangle}(p,\lambda)$, for a fixed λ and prime p, where $\langle w \rangle$ is the cyclic subgroup generated by w in Γ. The evaluation of our $\mathcal{L}$ function is complete for ι, τ, $\iota\tau$, and $\eta\iota$.
- Our research is intended to prove that the group Γ is transitive on the κ varieties off a set of arithmetic measure 0. Secondary questions we intend to answer involve classifying all the orbit types and classify the asymptotics of various subgroups of Γ.

Introduction

- The Markoff type Diophantine equation used here is $\kappa(x,y,z) = x^2 + y^2 + z^2 - xyz - 2$
- We consider κ with inputs from finite fields; this means that that collection of points (x,y,z) can be thought of as a cube of points.
- There is a group of polynomial κ-automorphisms, Γ, generated by three elements ι, τ and η given by the following maps (define the maps):

$$\tau\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}y\\x\\z\end{pmatrix} \qquad \iota\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\xy-z\end{pmatrix} \qquad \eta\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}z\\y\\yz-x\end{pmatrix}$$

- ι Is called a **Vieta involution** and η is a Vieta involution on the second coordinate followed by a permutation on the coordinates.
- For any $G \leq \Gamma$ we define $\mathcal{L}_G(p,\lambda)$ as $\max\{|Orb_G(v)| : v \in \kappa^{-1}(\lambda)\}$
- We define

$$\mathcal{L}_G^{max}(p) := \max\{\mathcal{L}_G(p,\lambda) : \lambda \in \mathbb{Z}_p\}$$
$$\mathcal{L}_G^{avg}(p) := \frac{1}{p}\sum_{\lambda \in \mathbb{Z}_p} \mathcal{L}_G(p,\lambda)$$

The Γ Ergodicity Conjecture

- Given a prime field, $\mathbb{Z}_p$ and any κ variety we suspect that most of the points in the variety will be a part of the same orbit.
- Formally put, since the size of the κ varieties is asymptotic to p^2, we intended to show:

$$\lim_{p\to\infty} \frac{\mathcal{L}_\Gamma^{max}(p)}{p^2} = 1$$

The $p\log(p)$ Conjecture

- We suspect that of all the elements of Γ $\eta\tau$ and its conjugates produce the fastest growing $\mathcal{L}_{\langle\varphi\rangle}^{max}$ and $\mathcal{L}_{\langle\varphi\rangle}^{avg}$ functions
- We also suspect that the growth rate for the the $\mathcal{L}$ functions mentioned is $p\log p$

The Orbit Classification Conjecture

- We suspect that given some prime field $\mathbb{Z}_p$ and any κ variety except $\kappa = 2$ that every orbit falls into one of 7 orbit types (each occurring at most once per variety):
  1. $\mathrm{Orb}_\Gamma((0,0,0))$ of order 1 when $\kappa = -2$
  2. $\mathrm{Orb}_\Gamma((0,0,x))$ of order 6 for all $x \in \mathbb{Z}_p^*$ when $\kappa + 2$ is a perfect square in $\mathbb{Z}_p$
  3. $\mathrm{Orb}_\Gamma((0,1,1))$ of order 16 when $\kappa = 0$
  4. $\mathrm{Orb}_\Gamma((0,1,\pm\sqrt{2}))$ of order 36 when $p \equiv \pm 1 \mod 8$ and $\kappa = 1$
  5. $\mathrm{Orb}_\Gamma((0,1,\frac{1\pm\sqrt{5}}{2}))$ of order 40 when $p \equiv \pm 1 \mod 10$ and $\lambda = \frac{1\pm\sqrt{5}}{2}$
  6. $\mathrm{Orb}_\Gamma((1,1,\frac{1\pm\sqrt{5}}{2}))$ of order 72 when $p \equiv \pm 1 \mod 10$ and $\kappa = 1$
  7. All other points in a given variety form a single orbit
- If this can be shown to be true then the ergodicity conjecture is true.

Orbits Visualized

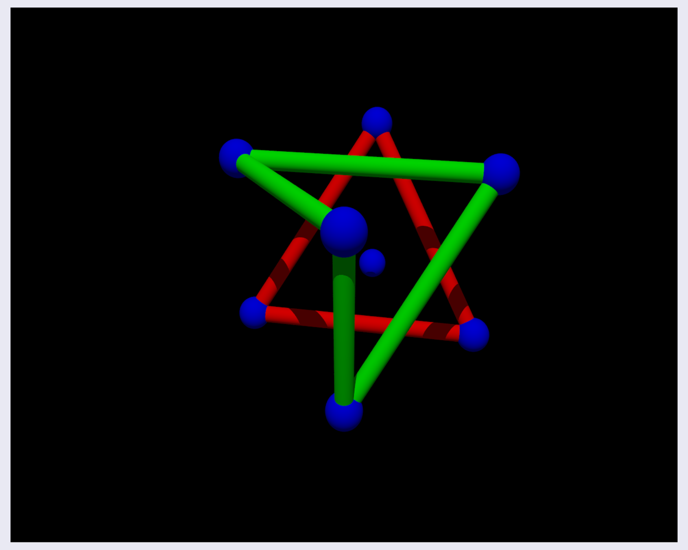

Figure: $\eta\tau$ orbits in $\mathbb{Z}_2^3$

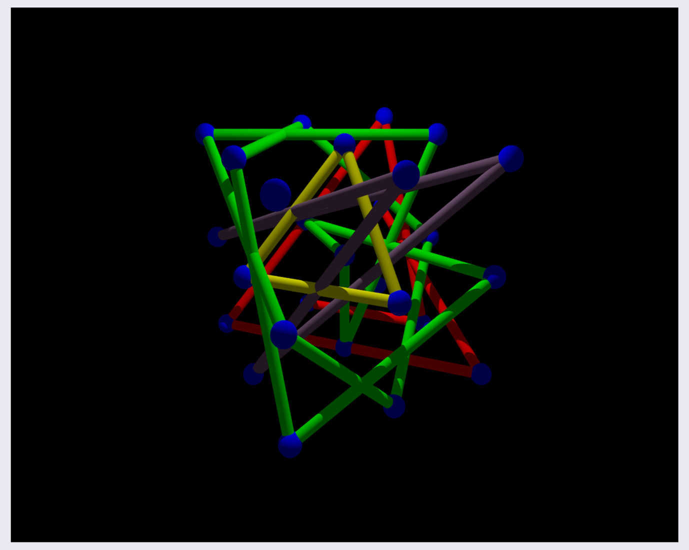

Figure: $\eta\tau$ orbits in $\mathbb{Z}_3^3$

Structure of Γ

- One of our key findings is that Γ is completely generated by η and τ. ι is necessary in a higher perspective of this problem (it is a required generator of $\mathrm{Out}(F_2)$) but its induced action on the cube of points can be mimicked by η and τ
- The relation is given by $\tau\eta\tau\eta^{-1}\tau\eta$
- This greatly simplifies the analysis of this problem, as we need only understand the behavior of τ, η, and the interactions between them

Facts about η

The behavior of τ is very well understood (it just switches the first and second coordinates), but η has rich structure.

- η fixes the second coordinate. Furthermore, if one considers a fixed y value, it acts as a linear transformation on the remaining coordinates. The transformation has the corresponding matrix:

$$\begin{pmatrix}0 & 1\\ -1 & y\end{pmatrix}$$

- Through diagonalization, Jordan normal form, and Galois theory we can understand everything about the orbit sizes of η. As an example, given the prime field $\mathbb{Z}_p$ we know that all η orbits have a size that must divide $2p$, $p+1$, or $p-1$.
- Using the techniques above we can show that if given a divisor of $2p, p+1$, or $p-1$ we can find an orbit of that order. This allows us to show that the order of η is $p\frac{p^2-1}{2}$
- We have shown that in an η orbit of size p or $2p$ every possible x coordinate value shows up.
- We currently do not understand the interaction between η and τ very well (for example we currently have no techniques to analyze $\eta\tau$ despite many attempts to understand it). The only algebraic relation we have found between them is $\tau\eta\tau = \eta^2\tau\eta^{-1}\tau\eta^2$

Computation

We use Python and Mathematica to create implementations of of the orbit structures discussed. We have designed programs that perform the following:

- Compute the orbit of any finitely generated subgroup of Γ
- Symbolically compute compositions of the generating functions
- A sequence asymptotics classification algorithm (returns a function that has similar end behavior as the input sequence)
- Visualization of orbits and orbit structure

Putting these algorithms together allow us to find exceptional orbits, count the number of orbits per variety, categorize elements of Γ by end behavior, find nontrivial identities in Γ and more.

Progress on $p\log(p)$ Conjecture

Our progress on the $p\log(p)$ conjecture is based entirely on numerical evidence

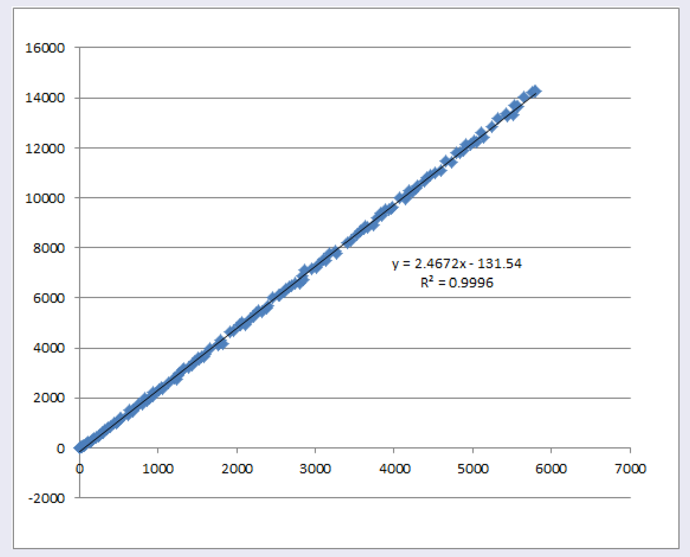

Figure: $\mathcal{L}_{\langle\eta\tau\rangle}^{avg}$ graphed against $p\log p$. The strong linear fit supports the $p\log p$ conjecture

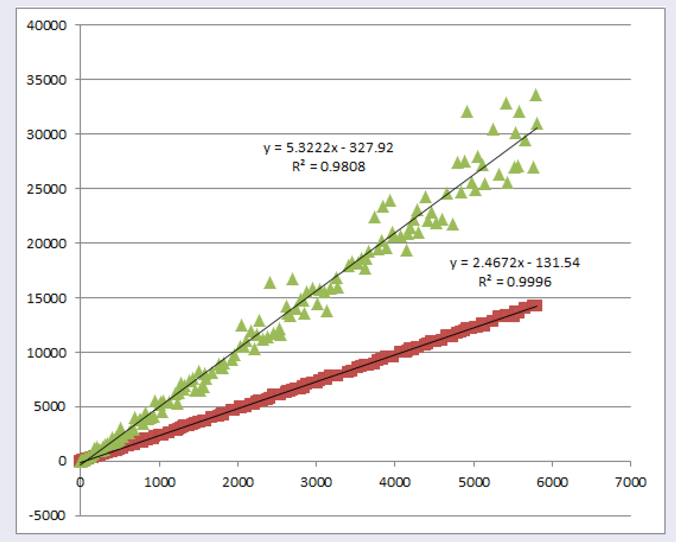

Figure: $\mathcal{L}_{\langle\eta\tau\rangle}^{avg}$ and $\mathcal{L}_{\langle\eta\tau\rangle}^{max}$ graphed against $p\log p$. The max variant appears to not be as well behaved as the average, but still approximately has $p\log p$ growth

As seen in the figures, there is a clear correlation between $p\log(p)$ and the growth of the $\eta\tau$ orbits. We currently do not have tools that seem capable of answering this conjecture.

Work on Orbit Classification Conjecture

- As stated before, if this conjecture holds then so does arithmetic ergodicity.
- We know that if we start from a point that generates a size p or $2p$ orbit then we can choose any x value we wish.
- We are currently looking for techniques that will enable us to reach any y value in addition to control the third.
- We are looking for a way to to think of finite field points as being in the algebraic closure of the rationals so that we can compare points between fields that may not have the same characteristic
- If we can find such relation then we can narrow our search of possible exceptional orbits. In $\overline{\mathbb{Q}}$ we know that exceptional orbits must lie in a topological ball contained inside the ball of infinity norm 2 centered at the origin.

Future Work

- We will keep searching for techniques that will connect any two points in a κ variety off the exceptional orbits.
- Recent computations have shown that the union of $\eta^k\tau$ orbits give the full orbit. We will begin studying these orbits and hopefully find a pattern or algorithm to determine which $\eta^k\tau$ orbit to use to connect any two points in a variety.
- We continue to search for a proof that the list of orbits in the Orbit Classification Conjecture is exhaustive and continue the search for exceptional orbits in case it is not exhaustive.
- We will attempt to work from the free group perspective (not discussed here) to analyze the case when $\kappa = 2$
- We continue to find more relationships between η and τ. If possible we will search for a presentation that gives a full description of the induced action of Γ on a given finite field.
- We will continue to generalize our results to the case when general finite fields are considered.

Acknowledgements

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