Computing Test Ideals of Cohen-Macaulay Modules

Julian Benali, Shrunal Pothagoni

under Dr. Rebecca R.G. at
George Mason University, MEGL

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Preliminaries

All rings we use are commutative, unital, Noetherian, and local. We will use k to denote a field, R to denote a ring, and M to denote a module. We will be working primarily with subrings and quotients of power series rings.

Definitions

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- A sequence $x_1, ..., x_n$ of elements in R is said to be a regular sequence over M if $(x_1, ..., x_n)M \neq M$ and x_i is not a zero divisor in $M/(x_1, ..., x_{i-1})M$ for all $i \in \{1, ..., n\}$.
  In other words, $(x_1, ..., x_n)M \neq M$ and for all $z \in M$, if $z \notin (x_1, ..., x_i)M$ then $x_{i+1}z \notin (x_1, ..., x_i)M$ as well.

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- We say M is a Cohen-Macaulay (CM) module over R if the length of the longest regular sequence over M is the same as the Krull dimension of R. A finitely generated CM module is called a maximal Cohen-Macaulay (MCM) module.

Definitions

- If R is a complete local domain, then for any R-module, M, one can show that the test ideal of M is

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- Our goal is then to compute the intersection of the test ideals of the MCM modules over R, denoted $\tau_{MCM}(R)$.

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- Every MCM module is a direct sum of indecomposable MCM modules. Furthermore, it follows from the definition that for any CM R-modules, N and L, $\tau_{N \oplus L}(R) = \tau_N(R) + \tau_L(R)$.

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From these facts, we conclude that $\tau_{MCM}(R)$ is the intersection of the test ideals of the non-free indecomposable MCM R-modules.

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For any CM module M over a regular local ring R, $\tau_M(R) = R$*, and hence,* $\tau_{MCM}(R) = R$.

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Given the MCM modules, we know how to compute their test ideals, and therefore, the intersection. This intersection gives an upper bound for the intersection of the test ideals of all CM modules.

Example (One-Dimensional Type A_n)

Let $R = k[\![x, y]\!]/(y^2 + x^{n+1})$ for a field k an even nonnegative integer, n. It can be shown that every non-free MCM R-module is isomorphic to

$$M_j = \operatorname{cok} \begin{pmatrix} y & x^{n+1-j} \\ -x^j & y \end{pmatrix}$$

for some positive integer $j \leq \frac{n}{2}$ (Yoshino, pg. 39). We showed $\tau_{M_j}(R) = (x^j, y)$ and thus,

$$\tau_{MCM}(R) = \cap \tau_{M_j}(R) = (x^{\frac{n}{2}}, y)$$

Illustration (One-Dimensional Type A_2)

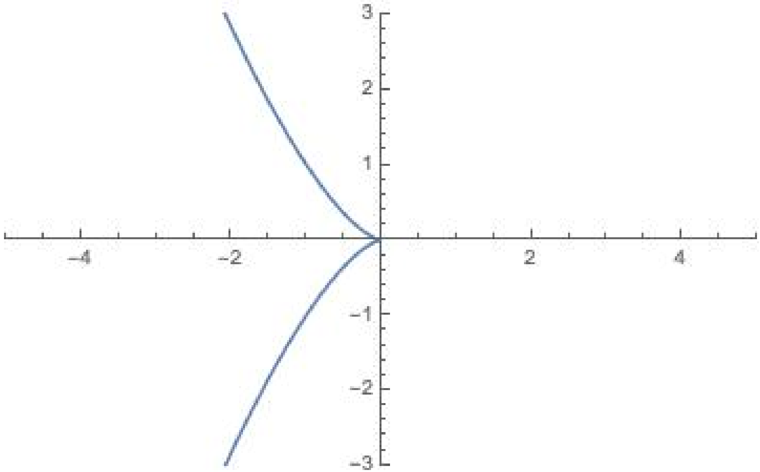

Figure: Plot of $y^2 + x^3 = 0$

Example (Two-Dimensional Type A_n)

Let $R = k[\![x, y, z]\!]/(x^2 + y^{n+1} + z^2)$ for an algebraically closed field k whose characteristic is not 2, 3, or 5. It can be shown that every non-free MCM R-module is isomorphic to

$$M_j = \operatorname{cok} \begin{pmatrix} z - ix & -y^{n+1-j} \\ y^j & z + ix \end{pmatrix}$$

for some positive integer $j \leq n$ (Leuschke and Wiegand, pg. 154). We showed $\tau_{M_j}(R) = (x^2, y^{\min\{j,n+1-j\}}, z^2)$ and thus,

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Illustration (Two-Dimensional A_4)

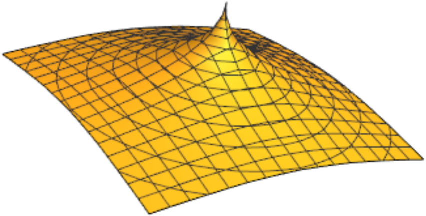

Figure: Plot of $z^2 + x^2 + y^5 = 0$

Results

Theorem (Benali and Pothagoni's Last Theorem 2019)

Suppose $R = k[\![x, y, z]\!]/(z^2 + g(x, y))$*. Let* φ *be an* $n \times n$ *matrix over* $k[\![x, y]\!] \subset k[\![x, y, z]\!]$ *and suppose* $(zI - \varphi, zI + \varphi)$ *is a matrix factorization for* $z^2 + g(x, y)$ *(i.e.* $(zI - \varphi)(zI + \varphi) = (z^2 + g(x, y))I$*). If* $M = \operatorname{cok}(\overline{zI - \varphi})$*, then*

$$Im(Hom_R(M, R) \hookrightarrow Hom_R(R^n, R) \xrightarrow{\cong} R^n) = Im(\overline{zI + \varphi^T}).$$

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Corollary

Let R, φ, *and* M *be as in the theorem. Then* $\tau_M(R)$ *is generated by the entries of the matrix* $\overline{zI + \varphi}$.

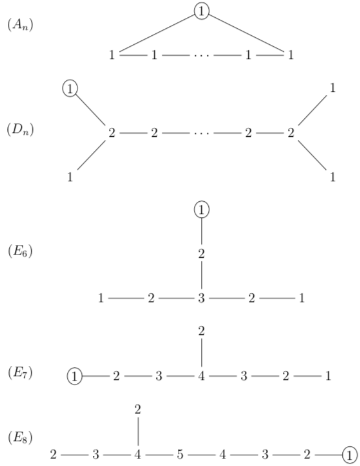
ADE Singularities

The ADE classification of singularities is a way of corresponding certain Dynkin Diagrams to singularites. The two-dimensional ADE singularities are given by

$$(A_n) : z^2 + x^2 + y^{n+1}$$
$$(D_n) : z^2 + x^2 y + y^{n-1}$$
$$(E_6) : z^2 + x^3 + y^4$$
$$(E_7) : z^2 + x^3 + xy^3$$
$$(E_8) : z^2 + x^3 + y^5$$

Using the theorem, we can immediately derive the intersection of the test ideals for these singularities and several others.

Dynkin Diagrams for ADE Singularities



This illustration shows ADE diagrams. These diagrams correspond to simply laced Dynkin Diagrams, a form of the ADE classification. (Leuschke and Wiegand pg. 92).

Conclusions

We proved a theorem which allows for easy computation of the test ideals of certain modules. We used this theorem to give a proof for our computations of the test ideals of ADE singularities where a direct proof would have been impractical.

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For example, this lead to the interesting result that the intersection of the test ideals for E_6, E_7, and E_8 were (x, y^2, z), (x, y^3, z), and (x, y^2, z) respectively.

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