Test Ideals of Cohen-Macaulay Modules

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Abstract

In our research, we compute the test ideals of finitely-generated CM (MCM) modules of various rings. The rings we study include subrings and quotients of polynomial rings and power series rings. The test ideals of such rings yield information about the rings' singularities and geometric properties: vaguely, the larger the test ideal, the less singular the ring, and vice versa. If the ring is nonsingular, the test ideal coming from any MCM module is the whole ring.

Introduction

All rings are assumed to be commutative, unital, Noetherian, and local. We will use R to denote a ring, m to denote its maximal ideal, and M to denote an R-module.

- The Krull dimension of R is the length of a longest proper chain of prime ideals $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n$.
- A sequence $x_1, \ldots, x_n$ of elements in R is said to be a regular sequence over M if $(x_1, \ldots, x_n)M \neq M$ and x_i is not a zero divisor in $M/(x_1, \ldots, x_{i-1})M$ for all $i \in \{1, \ldots, n\}$.
- We say M is a Cohen-Macaulay (CM) module over R if the length of a longest regular sequence over M is the same as the Krull dimension of R. A finitely generated CM module is called a maximal Cohen-Macaulay (MCM) module.

The Test Ideal

If R is a complete local domain, the test ideal of M is

$$\tau_M(R) = \sum_{f \in \mathrm{Hom}_R(M,R)} f(M)$$

That is, we find the images of the R-module homomorphisms from M to R and take their sum [Pérez–RG 2019]. Our goal is then to compute the intersection of the test ideals of the MCM modules, which we denote $\tau_{MCM}(R)$.

Computing Test Ideals

In practice, we do not need to compute the test ideal of every MCM R-module in order to compute $\tau_{MCM}(R)$.

- Let M be a nonzero free R-module. Then any projection map from M to R is a surjective R-module homomorphism and thus, $\tau_M(R) = R$.
- Every MCM module is a direct sum of indecomposable MCM modules. Furthermore, it follows from the definition that for any CM R-modules, N and L, $\tau_{N \oplus L}(R) = \tau_N(R) + \tau_L(R)$.

From these facts, we conclude that $\tau_{MCM}(R)$ is the intersection of the test ideals of the non-free indecomposable MCM R-modules.

Test Ideals of the Kleinian Singularities

Type A_n $(n \geq 1)$

Let $R = k[\![u^{n+1}, uv, v^{n+1}]\!]$. Up to isomorphism, the non-free indecomposable MCM modules are

$$M_j := R(u^a v^b \mid b - a \equiv j \mod n+1) = u^{n+1-j}R + v^j R$$

We showed that if $1 \leq j \leq \frac{n+1}{2}$, then

$$\tau_{M_j}(R) = R(u^{n+1}, u^j v^j, v^{n+1})$$

and if $\frac{n+1}{2} \leq j \leq n$, then

$$\tau_{M_j}(R) = R(u^{n+1}, u^{n+1-j} v^{n+1-j}, v^{n+1})$$

Consequently,

$$\tau_{MCM}(R) = R(u^{n+1}, u^{\lfloor \frac{n+1}{2} \rfloor} v^{\lfloor \frac{n+1}{2} \rfloor}, v^{n+1})$$

Types E_6, E_7, and E_8

Let

$$R_6 = k[\![x, y, z]\!]/(z^2 + x^3 + y^3)$$
$$R_7 = k[\![x, y, z]\!]/(z^2 + x^3 + xy^3)$$
$$R_8 = [\![x, y, z]\!]/(z^2 + x^3 + y^5)$$

The indecomposable MCM modules for these singularities are cokernels of matrices whose sizes range from 2×2 to 12×12. We showed that

$$\tau_{MCM}(R_6) = (x, y^2, z)$$
$$\tau_{MCM}(R_7) = (x, y^3, z)$$
$$\tau_{MCM}(R_8) = (x, y^2, z)$$

Type D_n $(n \geq 4)$

Let $R = k[\![x, y, z]\!]/(z^2 + x^2 y + y^{n-1})$ where n is odd and $n \geq 4$. The indecomposable MCM modules of R can be expressed as the cokernels are certain matrices. Let

$$\varphi_1 = \begin{pmatrix} z & x^2 + y^{n-2} \\ -y & z \end{pmatrix}$$

For $j \in \{2, 3, \ldots, n-2\}$, if j is even, let

$$\varphi_j = \begin{pmatrix} z & 0 & xy & y^{n-1-j/2} \\ 0 & z & y^{j/2} & -x \\ -x & -y^{n-1-j/2} & z & 0 \\ -y^{j/2} & xy & 0 & z \end{pmatrix}$$

The matrix is similar in the case that j is odd. Let

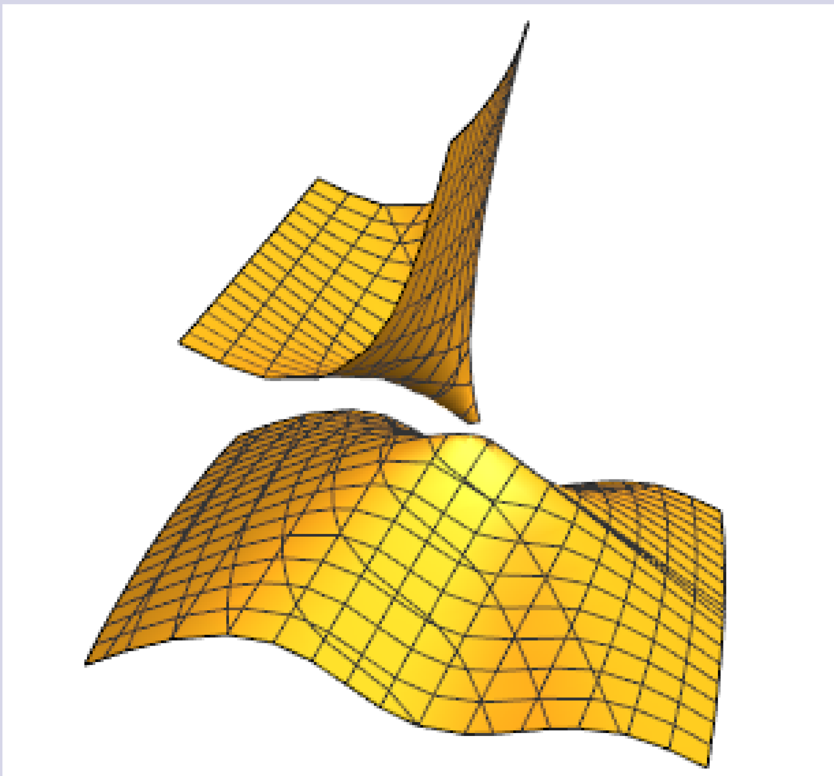
$$\varphi_{n-1} = \begin{pmatrix} z - iy^{(n-1)/2} & x \\ -xy & z + iy^{(n-1)/2} \end{pmatrix}$$

$$\varphi_n = \begin{pmatrix} z - iy^{(n-1)/2} & xy \\ -x & z + iy^{(n-1)/2} \end{pmatrix}$$

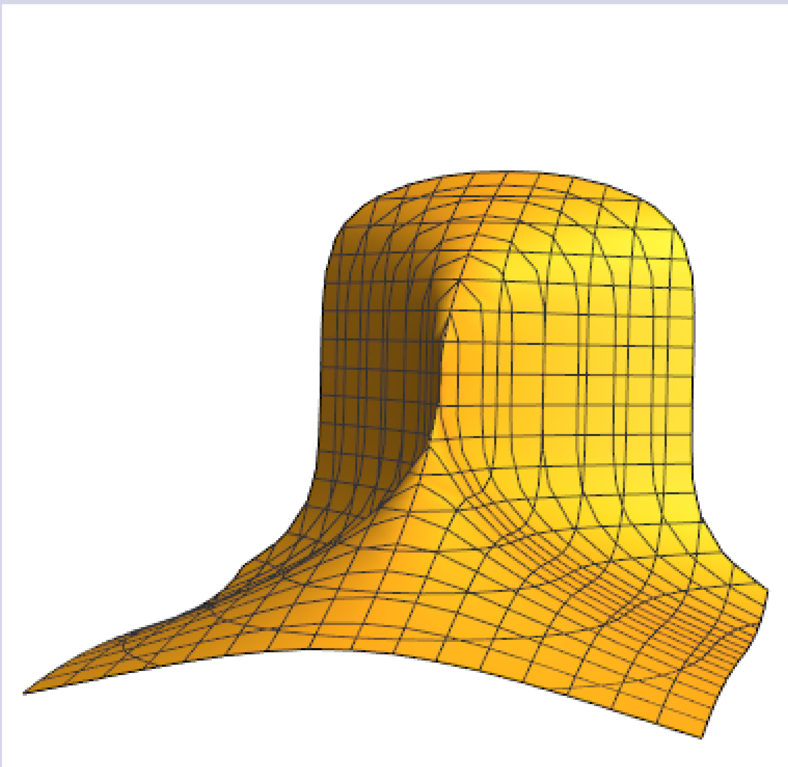
The matrices are similar in the case that n is even. For every $j \leq n$, let M_j be the cokernel of φ_j. We showed that $\tau_{M_1}(R) = (x^2, y, z)$ and that for $j > 1$, $\tau_{M_j} = (x, y^{\lfloor j/2 \rfloor}, z)$. Consequently,

$$\tau_{MCM}(R) = (x^2, y^{\lfloor n/2 \rfloor}, z)$$

Illustrations



Plot of the singularity E_7



Plot of the singularity E_8

Derivations of the Kleinian Singularities

The Kleinian singularities are two-dimensional hypersurfaces corresponding to ADE Dynkin Diagrams. We will use k to denote an algebraically closed field whose characteristic is not 2, 3, or 5. Given a suitable finite subgroup G of $GL(2, k)$, G acts on $k[\![x_1, x_2]\!]$ by linear changes of variables. The ring of invariants, R, of this group action is a Kleinian singularity. Furthermore, the indecomposable MCM R-modules will be in one-to-one correspondence with irreducible k-representations of G. Leuschke and Wiegand use this method to derive the indecomposable MCM modules of the Kleinian singularities on page 153 of *Cohen-Macaulay Representations*, which we then used to compute their test ideals.

Theorem (Benali and Pothagoni 2019)

Let $R = k[\![x, y, z]\!]/(z^2 + g(x, y))$ and let φ be a matrix over $k[\![x, y]\!]$ with 0 in its diagonal entries such that

$$(zI - \varphi)(zI + \varphi) = (z^2 + g(x, y))I$$

If M is the cokernel of $zI - \varphi$ as a matrix over R, then $\tau_M(R)$ is generated by the entries of $zI - \varphi$.

Consequences of the Theorem

The theorem allows us to easily compute the test ideals of any module of the form stated in the theorem, such as the indecomposable MCM modules for Kleinian singularities. Without the theorem, the computations, and their corresponding proofs, become exponentially more tedious with respect to the size of the presentation matrix of a module. Proving the computations for the type E_6, E_7, and E_8 singularities was therefore not previously feasible.

Conclusions

Our computations improve our understanding of the singularities we studied, and lead to some fascinating results. For example, we discovered that τ_{MCM} for E_6, E_7, and E_8 were (x, y^2, z), (x, y^3, z), and (x, y^2, z) respectively. This suggests that the singularity for E_7 is notably more severe than that of E_6 and E_8 since τ_{MCM} for E_6 and E_8 is contained in a smaller power of the maximal ideal.

References

- Graham J. Leuschke and Roger Wiegand. Cohen-macaulay representations. In preparation. http://www.leuschke.org/Research/MCMBook
- Felipe Pérez and Rebecca R.G. Characteristic-free test ideals. arXiv:1907.02150, preprint, 2019.