Problems in Discrete Geometry Using SAT Solvers

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Overview

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  - Translate problems for SAT solvers
  - Verify existing bounds and human proofs
  - Discover unknown values of $N_d(n)$
- Motivation
  - Approachable discrete geometry problems
  - Problems are well-suited for computation
  - Very recent advances in the field

Points in Convex Position

- What is the smallest number $N_d(n)$ so that any set of $N_d(n)$ points in general position in $\mathbb{R}^d$ contains n points in convex position?

- A set of points is in convex position if none of the points is a convex combination of the others.

- A point x is a convex combination of points in $\{e_1, ..., e_k\}$ in $\mathbb{R}^d$ if $x = \sum_{i=1}^{k} \lambda_i e_i$ where $\sum_{i=1}^{k} \lambda_i = 1$ and $\lambda_i \geq 0$ for all i.

Esther Klein Problem

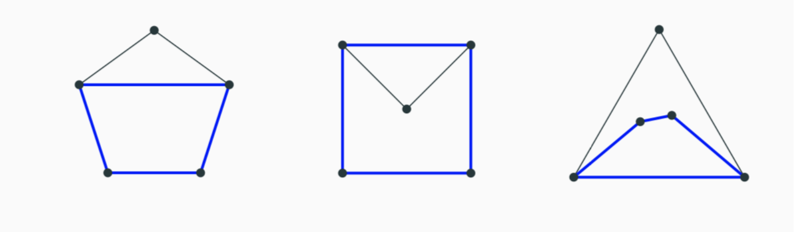


Figure: Configurations of 5 points in General Position

Chirotopes

- A chirotope realized by E is the function χ mapping a set of ordered $(d+1)-$element subsets of points in $E = \{e_1, e_2, ..., e_k\}$ to the set $\{-1, 1\}$.
- For example, in 2D space consider the points $e_1 = (x_1, y_1)$, $e_2 = (x_2, y_2)$, and $e_3 = (x_3, y_3)$,
- $\chi(\{e_1, e_2, e_3\}) =$sgn det$(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ where $\hat{e}_i = (x_i, y_i, 1)$.
- If $\chi(\{e_1, e_2, e_3\}) = -1$, we traverse the triangle formed by the points e_1, e_2, e_3 clockwise.
- If $\chi(\{e_1, e_2, e_3\}) = 1$, we traverse the triangle formed by the points e_1, e_2, e_3 counterclockwise.

Chirotopes Continued

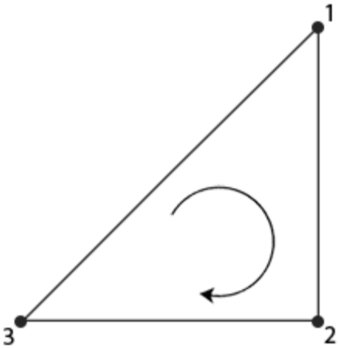


Figure: Negative vs. Positive Determinant

Grassmann Plücker Relations

- Let $E = \{e_1, e_2, ..., e_n\}$. The χ values of these points must satisfy the Grassmann Plücker relations.
- $\{-1, 1\} \subseteq \{\chi(\sigma, e_1, e_2)\chi(\sigma, e_3, e_4), -\chi(\sigma, e_1, e_3)\chi(\sigma, e_2, e_4), \chi(\sigma, e_1, e_4)\chi(\sigma, e_2, e_3)\}$ for all σ in $\binom{n}{d-1}$ and $\{e_1, e_2, e_3, e_4\} \subseteq E \backslash \sigma$.
- This will generate $\binom{n}{d-1}\binom{n-d+1}{4}$ clauses for a given n and d.

Acyclicity

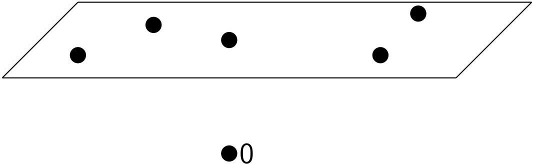


Figure: Point set in a plane above the origin

Acyclicity

- We want to make sure that our set of points will not contain any positive circuits.
- In $d = 2$, a circuit is a vector x satisfying $Ax = 0$ where x has 4 nonzero entries.
- For a set of 4 points, let $\hat{e}_1 x_1 + \hat{e}_2 x_2 + \hat{e}_3 x_3 + \hat{e}_4 x_4 = 0$.
- By Cramer's Rule, we get that $-\frac{x_1}{x_4} = \frac{det(\hat{e}_4, \hat{e}_2, \hat{e}_3)}{det(\hat{e}_1, \hat{e}_2, \hat{e}_3)}$, $-\frac{x_2}{x_4} = \frac{det(\hat{e}_1, \hat{e}_4, \hat{e}_3)}{det(\hat{e}_1, \hat{e}_2, \hat{e}_3)}$, $-\frac{x_3}{x_4} = \frac{det(\hat{e}_1, \hat{e}_2, \hat{e}_4)}{det(\hat{e}_1, \hat{e}_2, \hat{e}_3)}$.
- If there is a positive circuit, $\chi(\{e_1, e_2, e_3\}) = -\chi(\{e_1, e_2, e_4\}) = \chi(\{e_1, e_3, e_4\}) = -\chi(\{e_2, e_3, e_4\})$.

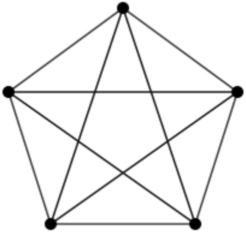
CNF & SAT Solvers

- Conjunctive Normal Form
  - Conjunction of clauses
  - Each clause is disjunction of literals
  - Each literal is a Boolean variable or its negation
  - Each variable is a chirotope
    - Sign of the determinant
    - Pairwise coloring

$$(\chi(\{e_1, e_2, e_3\}) = +1 \vee \chi(\{e_1, e_2, e_4\}) = -1 \vee \chi(\{e_1, e_3, e_4\}) = +1 \vee \chi(\{e_2, e_3, e_4\}) = -1)$$
$$\wedge(\chi(\{e_1, e_2, e_3\}) = -1 \vee \chi(\{e_1, e_2, e_4\}) = +1 \vee \chi(\{e_1, e_3, e_4\}) = -1 \vee \chi(\{e_2, e_3, e_4\}) = +1)$$

- Satisfiability Solvers
  - NP-complete
  - Systematic backtracking search
  - Many available via open-source, eg: Glucose

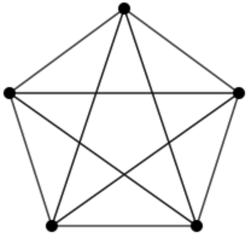
Ramsey Theory: $R_2(3, 3)$



Need to find 2-coloring of the set of pairs so that:

no three pairs in the same triangle have the same color.

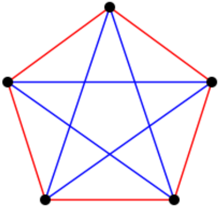
Ramsey Theory: $R_2(3,3)$



For every 3-element subset (i, j, k)

$$(\chi(i,j) = 1 \vee \chi(i,k) = 1 \vee \chi(j,k) = 1)$$
$$\wedge(\chi(i,j) = -1 \vee \chi(i,k) = -1 \vee \chi(j,k) = -1)$$

Ramsey Theory: $R_2(3,3)$



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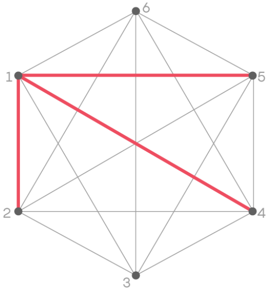
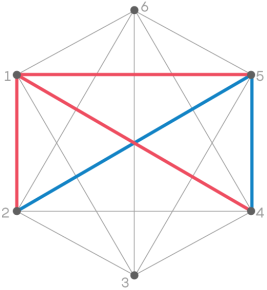


$$\chi(1,2) = 1 \wedge \chi(1,4) = 1 \wedge \chi(1,5) = 1$$

Ramsey Theory: $R_2(3,3)$



$$(\chi(1,2) = 1 \vee \chi(1,4) = 1 \vee \chi(2,4) = 1)$$

$$\wedge(\cancel{\chi(1,2) = -1} \vee \cancel{\chi(1,4) = -1} \vee \chi(2,4) = -1)$$

$$(\chi(2,4) = 1 \vee \cancel{\chi(2,5) = 1} \vee \cancel{\chi(4,5) = 1})$$

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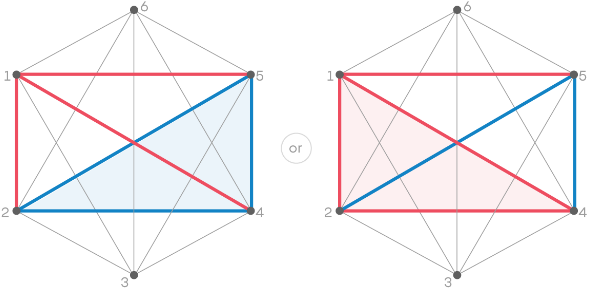
Ramsey Theory: $R_2(3,3)$



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Erdős on Computational Complexity

"Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for **red five** and **blue five**. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack."

- $R_2(5,5)$
  - $43 \leq R_2(5,5) \leq 48$
  - $\binom{43}{2} = 903$ pairwise relationships
  - $2^{903} \approx 10^{272}$ possible configurations

Erdős on Computational Complexity

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- $R_2(6, 6)$
  - $102 \leq R_2(6, 6) \leq 165$
  - $\binom{102}{2} = 5151$ pairwise relationships
  - $2^{5151} \approx 10^{1550}$ possible configurations

Ramsey Theorem Results

- $R_n(r,s)$
  - n colors/relations
  - consider r-element and s-element subsets $(i, j, k, \ldots)$
  - $\binom{R_n(r,s)}{r}$ and $\binom{R_n(r,s)}{s}$ possible subsets
  - CNF with chirotopes of pairwise points
- Preliminary results
  - $R_2(3,3) \leq 6$
  - $R_2(4,4) \leq 18$
  - $R_2(3,4) \leq 9$

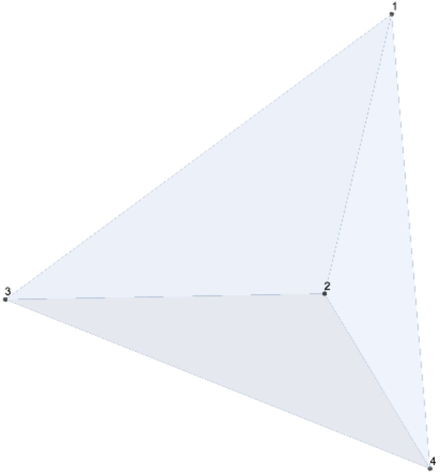
Revisiting Esther-Klein Results

- $N_2(n) \geq 2^{n+2} + 1$
  - Proven by Erdös and Szekeres in 1935
  - Useful to have explicit constructions of counter-examples
- $N_2(n) \leq 2^{n+o(n)}$
  - Proved by Suk in 2016
- $N_2(n) \stackrel{?}{=} 2^{n+2} + 1$
  - Conjecture of Erdös and Szekeres
  - $N_2(4) = 5$
  - $N_2(5) = 9$
  - $N_2(6) = 17$

Constraints & Extensions

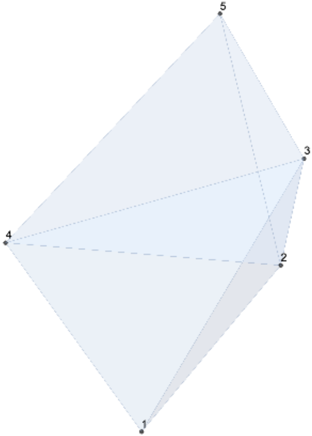
- Constraints
  - General position w.r.t. 3D space
  - Convexity
  - (A)cyclicity
  - Grassmann-Plücker
  - Categories of 3-polytopes
- Possible Extensions
  - More vertices
  - Higher dimensions
  - Searching for general formulas
  - General SAT solvers

General Position in 3D



$$N_3(4) = 4$$

Triangular Bipyramid



$$N_3(5) = 6$$

Octahedron

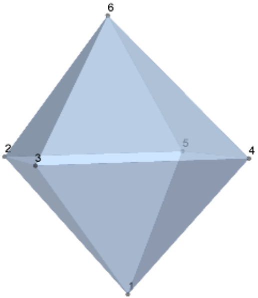


Figure: Octahedron in General Position

Cyclic Polytope

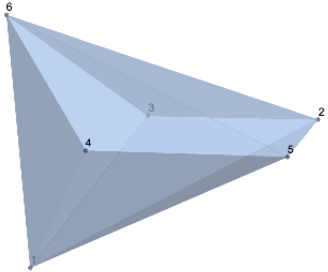


Figure: Cyclic Polytope in General Position

$$\chi(i, j, k, l) = 1 \text{ or -1} \quad \forall\ i < j < k < l$$

Known values of $N_d(n)$

		d = 2	3	4	5	6	7	…
n	3	3						
	4	5	4					
	5	9	6	5				
	6	17	9	7	6			
	7			9	8	7		
	8				10	9	8	
	9					11	10	
	10					13	12	
	…							

- Blue cells: upper bound proved by Bisztriczky and Harborth, lower bound by Morris and Soltan (human proof). $N_d(n) = 2n - d - 1$ for $d + 2 \leq n \leq \frac{3d}{2} + 1$
- $N_3(6)$ was found by Bisztriczky and Soltan
- Red cell: found by SAT solvers, no known human proof

Zoo of bounds

- $N_2(n) \leq 2^{n+o(n)}$ (proven in 2016)
- $N_2(k) \leq \binom{2k-5}{k-3} + 2$
- $N_2(k) \leq R_4(k, 5)$ and $N_2(k) \leq R_3(k, k)$
- $N_2(k) \geq 2^{n-2} + 1$ (conjectured to be equal)
- $N_d(k) \leq \binom{2k-2d-1}{k-d} + d$
- For each d there exists a c such that $N_d(k) = \Omega(c^{\sqrt[d-1]{k}})$

$\tilde{\tilde{N}}_d(k)$

- Like $N_d(k)$ but for different polytopes:
  - It's possible to reorder the points so that $\chi(\{e_1, ..., e_{d+1}\}) = \chi(\{e_1, ..., e_d, e_{d+2}\}) = ... = \chi(\{e_{k-d+1}, ..., e_k\}) = 1$ or -1 (ie $\chi(\{e_{r_1}, ..., e_{r_{d+1}}\}) = 1$ or -1 for all $r_1 < ... < r_{d+1}$)
- This guarantees convexity.
  - Assume that $\lambda_1 v_1 + ... + \lambda_{d+1} v_{d+1} = w$ where each $\lambda_i \geq 0$ and $\sum \lambda_i = 1$. You can rewrite this by

$$\begin{pmatrix} v_1 & v_2 & ... & v_{d+1} \\ 1 & 1 & ... & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ ... \\ \lambda_{d+1} \end{pmatrix} = \begin{pmatrix} w \\ 1 \end{pmatrix}.$$

    Apply Cramer's Rule and note that determinants are alternating. If the above condition is satisfied, you can work out the cases to find that one of $\lambda_1, ..., \lambda_{d+1}$ must be positive and one must be negative.
- Some examples of these polytopes are generated by sampling ordered points along the curve $(t, t^2, ..., t^d)^t$.

Background for $\tilde{\tilde{N}}_3(6)$

- $N_d(k) \leq \tilde{\tilde{N}}_d(k)$, but $\tilde{\tilde{N}}_d(k)$ is also significantly easier for a SAT solver to solve
- To exclude these polytopes, terms of the form

$$(\chi(\{e_1, ..., e_{d+1}\}) = 1 \vee \chi(\{e_1, ..., e_d, e_{d+2}\}) = 1 \vee \ldots \vee \chi(\{e_{k-d+1}, ..., e_k\}) = 1) \wedge$$
$$(\chi(\{e_1, ..., e_{d+1}\}) = -1 \vee \chi(\{e_1, ..., e_d, e_{d+2}\}) = -1 \vee \ldots \vee \chi(\{e_{k-d+1}, ..., e_k\}) = -1)$$

- For every pair (d, k), there will be at least 2 re-orderings that preserve the existence of positive and negative chirotope values, so this reduces the number of clauses by at least half
- This differs from the way to exclude general convex polytopes: excluding certain designated circuits, which involves exponential amounts of clauses in number of points
- SAT is NP-Hard in terms of the number of clauses, so this is good
- For 6 vertices, there are two "combinatorial types" of convex polehedra, the octohedron and the cyclic 8-polytope

$\tilde{\tilde{N}}_3(6)$

- Computation is still running!
- So far, counterexample chirotopes have been found for 9 and 10 point sets.
- However, the jury is still out on whether or not any of the 10-point chirotopes are realizable in 3D space. Finding a realization of a chirotope is a nonlinear problem.
- The counterexamples for 9 and 10 point sets were found in around 10 minutes, but case with 11 points has been running for the past couple of weeks.