Problems in Discrete Geometry Using Satisfiability Solvers

Daniel Taylor, Julia Rima, Sumanth Ravipati with Professor Dr. Walter Morris

Mason Experimental Geometry Lab

GEORGE MASON UNIVERSITY

December 10, 2018

Introduction

- Goals
  - Translate problems for SAT solvers
  - Verify existing bounds and human proofs
  - Discover unknown values of $N_d(n)$
- Motivation
  - Approachable discrete geometry problems
  - Problems are well-suited for computation
  - Very recent advances in the field

Definitions

Convex Position

- A set of points is in convex position if none of the points is a convex combination of the others.
- A point x is a convex combination of points in $\{e_1, ..., e_k\}$ in $\mathbb{R}^d$ if $x = \sum_{i=1}^{k} \lambda_i e_i$ where $\sum_{i=1}^{k} \lambda_i = 1$ and $\lambda_i \geq 0$ for all i.

Chirotope

- A chirotope realized by E is the function χ mapping a set of ordered $(d+1)-$element subsets of points in $E = \{e_1, e_2, ..., e_k\}$ to the set $\{-1, 1\}$. For example, in 2D space consider the points $e_1 = (x_1, y_1)$, $e_2 = (x_2, y_2)$, and $e_3 = (x_3, y_3)$,
- $\chi(\{e_1, e_2, e_3\}) =$sgn det$(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ where $\hat{e}_i = (x_i, y_i, 1)$.
- If $\chi(\{e_1, e_2, e_3\}) = -1$, we traverse the triangle formed by the points e_1, e_2, e_3 clockwise. If $\chi(\{e_1, e_2, e_3\}) = 1$, we traverse the triangle formed by the points e_1, e_2, e_3 counterclockwise.

Grassmann Plücker Relations

- Let $E = \{e_1, e_2, ..., e_n\}$. The χ values of these points must satisfy the Grassmann Plücker relations.
- $\{-1,1\} \subseteq \{\chi(\sigma, e_1, e_2)\chi(\sigma, e_3, e_4), -\chi(\sigma, e_1, e_3)\chi(\sigma, e_2, e_4), \chi(\sigma, e_1, e_4)\chi(\sigma, e_2, e_3)\}$ for all σ in $\binom{n}{d-1}$ and $\{e_1, e_2, e_3, e_4\} \subseteq E \backslash \sigma$.
- This will generate $\binom{n}{d-1}\binom{n-d+1}{4}$ clauses for a given n and d.

Acyclicity

- We want to make sure that our set of points will not contain any positive circuits. In $d = 2$, a circuit is a vector x satisfying $Ax = 0$ where x has 4 nonzero entries.
- For a set of 4 points, let $\hat{e}_1 x_1 + \hat{e}_2 x_2 + \hat{e}_3 x_3 + \hat{e}_4 x_4 = 0$.
- By Cramer's Rule, we get that $-\frac{x_1}{x_4} = \frac{det(\hat{e}_4, \hat{e}_2, \hat{e}_3)}{det(\hat{e}_1, \hat{e}_2, \hat{e}_3)}$, $-\frac{x_2}{x_4} = \frac{det(\hat{e}_1, \hat{e}_4, \hat{e}_3)}{det(\hat{e}_1, \hat{e}_2, \hat{e}_3)}$, $-\frac{x_3}{x_4} = \frac{det(\hat{e}_1, \hat{e}_2, \hat{e}_4)}{det(\hat{e}_1, \hat{e}_2, \hat{e}_3)}$.
- If there is a positive circuit, $\chi(\{e_1, e_2, e_3\}) = -\chi(\{e_1, e_2, e_4\}) = \chi(\{e_1, e_3, e_4\}) = -\chi(\{e_2, e_3, e_4\})$.

Esther Klein Problem

We want to find the smallest number $N_d(n)$ where $N_d(n)$ points in general position in $\mathbb{R}^d$ has n points in convex position. In two dimensional space, a set of points is in general position if no 3 points are on a line, and in three dimensional space, a set of points is in general position if no 4 points lie on the same plane. A set of points in convex position in two dimensional space will form the vertices of a convex polygon, and set of points in convex position in three dimensional space will form the vertices a convex polytope.

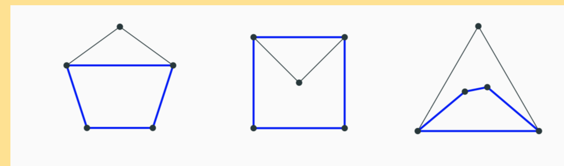


Figure: Configurations of 5 points in General Position

- $N_2(n) \geq 2^{n+2} + 1$, Erdös and Szekeres (1935)
- $N_2(n) \leq 2^{n+o(n)}$, Suk (2016)
- $N_2(n) \stackrel{?}{=} 2^{n+2} + 1$, Conjecture of Erdös and Szekeres
- $N_2(4) = 5$, $N_2(5) = 9$, $N_2(6) = 17$
- $N_2(n)$ is unknown for $n > 6$

Conjunctive Normal Form (CNF) & Satisfiability (SAT) Solvers

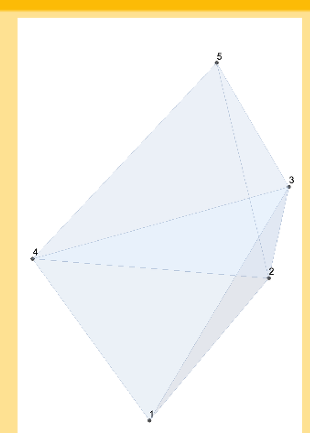
Conjunctive Normal Form is a conjunction of clauses, where each clause is disjunction of literals, each literal is a Boolean variable or its negation, and each variable is a chirotope. Examples of chirotpes include the sign of the determinant and pairwise coloring. Satisfiability solvers are NP-complete and use systematic backtracking search algorithms. We used an open-source implementation called Glucose, after generating clauses in Sage.

We can convert chirotopes conditions into CNF to feed into our SAT solver. For example, the previous acyclicity condition $\chi(\{e_1, e_2, e_3\}) = -\chi(\{e_1, e_2, e_4\}) = \chi(\{e_1, e_3, e_4\}) = -\chi(\{e_2, e_3, e_4\})$ can be written as the 2 clauses below:

$$(\chi(\{e_1, e_2, e_3\}) = +1 \vee \chi(\{e_1, e_2, e_4\}) = -1 \vee \chi(\{e_1, e_3, e_4\}) = +1 \vee \chi(\{e_2, e_3, e_4\}) = -1)$$
$$\wedge(\chi(\{e_1, e_2, e_3\}) = -1 \vee \chi(\{e_1, e_2, e_4\}) = +1 \vee \chi(\{e_1, e_3, e_4\}) = -1 \vee \chi(\{e_2, e_3, e_4\}) = +1)$$

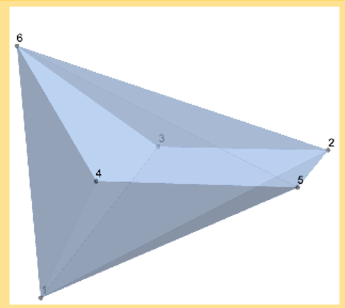
Extension to 3 Dimensions

$N_3(5) = 6$



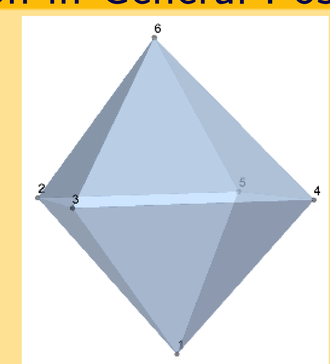
$\chi(i, j, k, l) = 1$ or -1 $\forall\ i < j < k < l$

Cyclic 3-Polytope in General Position



A Cyclic polytope, $C(n, d)$, is a convex polytope formed as a convex hull of n distinct points on a rational normal curve in R^d.

Octahedron in General Position

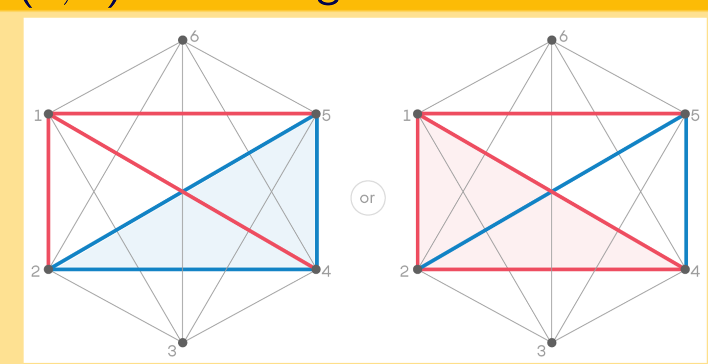


For 3 dimensions, and 6 vertices, it has been shown by Bisztriczky and Soltan that $N_3(5) = 6$.

Exploration of Ramsey Theory Problems

Ramsey's theorem states that there exists a least positive integer $R(r, s)$ for which every blue-red edge colouring of the complete graph on $R(r, s)$ vertices contains a blue clique on r vertices or a red clique on s vertices. There will be $\binom{R(r,s)}{r}$ and $\binom{R(r,s)}{s}$ possible subsets r and s subsets, respectively. This translates neatly to SAT solver clauses with chirotopes of pairwise points. We have independently found that $R(3, 3) = 6$, $R(4, 4) = 18$, and $R(3, 4) = 9$ and the search for $R(5, 5)$ is still underway.

Proof of $R(3, 3) = 6$ Using Clauses



False, if for every 3-element subset (i, j, k) :

$$(\chi(i, j) = 1 \vee \chi(i, k) = 1 \vee \chi(j, k) = 1)$$
$$\wedge(\chi(i, j) = -1 \vee \chi(i, k) = -1 \vee \chi(j, k) = -1)$$

Known Values for $N_d(n)$

n \ d	2	3	4	5	6	7	...
3	3						
4	5	4					
5	9	6	5				
6	17	9	7	6			
7			9	8	7		
8				10	9	8	
9					11	10	
10					13	12	
...							

- Blue cells: upper bound proved by Bisztriczky and Harborth, lower bound by Morris and Soltan (human proof). $N_d(n) = 2n - d - 1$ for $d + 2 \leq n \leq \frac{3d}{2} + 1$
- Red cell: found by SAT solvers, no known human proof

The Search for Cyclic Polytopes & Conclusions

$\tilde{N}_d(n)$ is the search for points in general position to guarantee n points as a cyclic d-polytope. $N_d(n) \leq \tilde{N}_d(n)$, but $\tilde{N}_d(n)$ is also significantly easier for a SAT solver to solve. To exclude these polytopes, we use terms of the form

$$(\chi(\{e_1, ..., e_{d+1}\}) = 1 \vee \chi(\{e_1, ..., e_d, e_{d+2}\}) = 1 \vee \ldots \vee \chi(\{e_{k-d+1}, ..., e_k\}) = 1) \wedge$$
$$(\chi(\{e_1, ..., e_{d+1}\}) = -1 \vee \chi(\{e_1, ..., e_d, e_{d+2}\}) = -1 \vee \ldots \vee \chi(\{e_{k-d+1}, ..., e_k\}) = -1)$$

For every pair (d, k), there will be at least 2 re-orderings that preserve the existence of positive and negative chirotope values, so this reduces the number of clauses by at least half. This differs from the way to exclude general convex polytopes: excluding certain designated circuits, which involves exponential amounts of clauses and number of points.

Future Work and Goals

- Computation is still running! So far, counterexample chirotopes have been found for 9 and 10 point sets.
- However, the jury is still out on whether or not any of the 10-point chirotopes are realizable in 3D space. Finding a realization of a chirotope is a nonlinear problem.
- The counterexamples for 9 and 10 point sets were found in around 10 minutes, but case with 11 points has been running since November 2018.

References

- Erds, P.; Szekeres, G. (1935), "A combinatorial problem in geometry", Compositio Mathematica, 2
- Morris, W.; Soltan, V. (2000), "The Erds-Szekeres problem on points in convex positionA survey", Bulletin of the American Mathematical Society, 37 (04)
- Szekeres, G.; Peters, L. (2006), "Computer solution to the 17-point Erds-Szekeres problem", ANZIAM Journal, 48 (02)
- Suk, Andrew (2016), "On the ErdsSzekeres convex polygon problem", J. Amer. Math. Soc.
- Conlon, David (2009), "A new upper bound for diagonal Ramsey numbers", Annals of Mathematics, 170 (2)