# Statistics in Deformations of Large Knots

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## Introduction

Goal: to discover trends in knots based on a measure of complexity, the Krull dimension of the character variety of the knot group.
We calculate dimension from a knot $K$ by first considering its complement, $K^C := \mathbb{R}^3 - K$. We then calculate the fundamental group. Next, we calculate the character variety. Finally, we calculate the Krull dimension.

$$K \implies K^C \implies \pi_1(K^C) \implies \mathcal{X}(\pi_1(K^C)) \implies \dim(\mathcal{X}(\pi_1(K^C)))$$

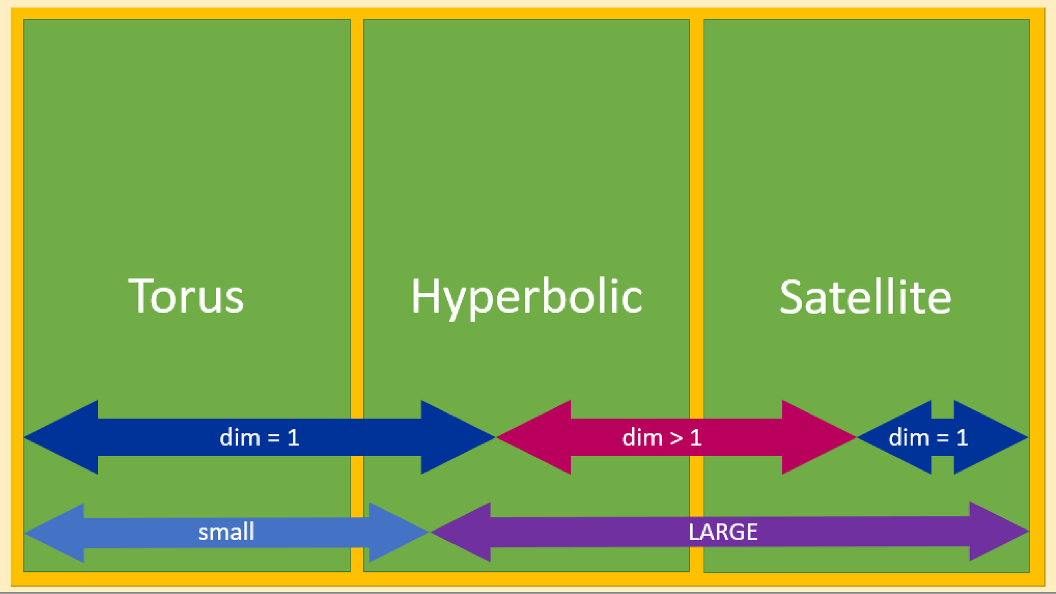
For sake of simplicity, we will denote $\dim(K) := \dim(\mathcal{X}(\pi_1(K^C)))$. Last semester's work has allowed us to refine our current work by classifying knots in a number of different ways; each classification has different properties that affect the calculations.

## Definitions

- Knot: An embedding of a circle, $S^1$, into $\mathbb{R}^3$.
- Fundamental Group: The group of base loops in a space up to equivalence
- Character Variety: Set of homomorphisms from the fundamental group to $\mathrm{SL}(2, \mathbb{C})$. A linear algebraic object used to describe specific types of groups.
- Krull Dimension: A measure of complexity of the ideal generated by the character variety
- Large Knot: A knot whose complement admits a closed essential surface.

## Classification of Knots

There are several lenses through which we can view knots. The first is the most common. Knots can be classifies as exactly one of the following: torus, hyperbolic or satellite. Next we can view them as large or small. Lastly, we can view them by their dimension. We combine these views in the graphic below.




## Conjectures

Given our previous results and the partitions of knots, we have the following conjectures:

Rank Conjecture: For satellite knots, there exists $r_0 \in \mathbb{N}$ such that if $\mathrm{rank}(\pi_1(K^c)) \geq r_0$, then $\dim(K) \geq 2$.

Volume Conjecture: For hyperbolic knots $K$, there exists $\alpha \in \mathbb{R}$ such that if $\mathrm{vol}(K^c) \geq \alpha$, then $\dim(K) \geq 2$ (This conjecture is stated in more detail later.)

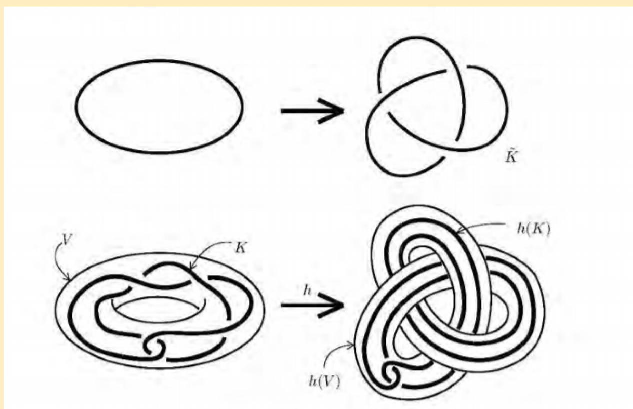
## Calculating Dimensions of Large Knots

Calculating the character varieties of resulting from large knots can be done for any knot using *SnapPy*, typically without much trouble. However, calculating the dimension of the character varieties is significantly more difficult. Dimension calculation in *Magma* is ideal since it utilizes a state-of-the-art algorithm. But even using this high-end algorithm, calculating the dimensions of nastier character varieties becomes unreasonable for most machines.
One possible solution is to utilize the ARGO cluster.
Ultimately, a "dream program" which inputs knots and outputs the dimensions would be ideal.

## Constructing Satellite Knots

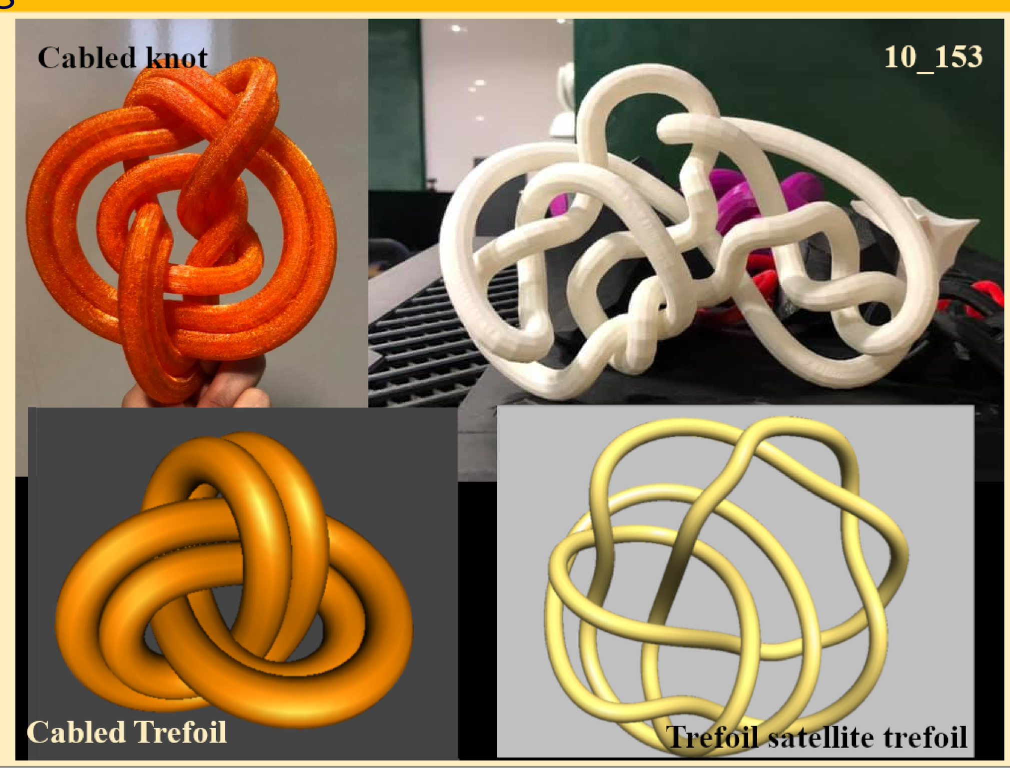
Satellite knots are formally constructed via the following method:

1. Begin with two knots, call them $A$ and $B$.
2. Nontrivialy embed $A$ into a filled torus.
3. "Tie" the filled torus into the shape of $B$.
4. The image of $A$ under this "tying" is a satellite knot.



Satellite Knot Construction [3]

## Visualizing Knots




## $\alpha$-conjecture

- Def: A tangle is a complement of two disjoint strands in $\mathbb{R}^3$.
- Def: A Montesinos knot is a knot that may be viewed as a composition of rational tangles in cyclic fashion.
- Def: A Montesinos knot of Kinoshita-Terasaka type is a Montesinos knot with at least four rational tangles, one of which has a defining reduced fraction of the form $\frac{a_i}{2b_i}$. [4]
- Theorem (Weak $\alpha$-Theorem): Let $K$ be a Montesinos knot of Kinoshita-Terasaka type with $n > 3$ tangles, at least 2 of which are positive and 2 of which are negative. Denote the knot complement by $M$. Then the $SL_2(\mathbb{C})$-character variety $\mathcal{X}(M)$ has complex dimension at least $n - 3$. Moreover, $K$ is a hyperbolic knot whose volume is bounded by the twist number $t(K)$:

$$\frac{v_8}{4}(t(K) - 9) \leq \mathrm{vol}(M) \leq 2v_8 t(K).$$

  Note that $v_8 \approx 3.6638$ is the volume of the regular ideal octahedron.
- Corollary: There exists a sequence of hyperbolic knots $\{K_n\}$ such that both $\mathrm{vol}(K_n)$ and $\dim_{\mathbb{C}}\mathcal{X}(K_n)$ can be arbitrarily large.
- Conjecture (Strong $\alpha$-Conjecture): Let $M$ be the knot complement of a hyperbolic knot. For every $k > 1$ there exists $\alpha_k > 0$ such that $\mathrm{vol}(M) \geq \alpha_k$ implies that $\dim_{\mathbb{C}}\mathcal{X}(M) \geq k$.

## Composite extension problem

Theorem: Let $K_1 \# K_2$ denote the connected sum of knots $K_1$ and $K_2$ (a composite knot made from $K_1$ and $K_2$).

$$\dim(K_1 \# K_2) \geq \max\{\dim(K_1), \dim(K_2)\}$$

- Composite knots are a subset of satellite knots.[1]
- This result generalizes inductively to connected sums of three or more knots.
- The knot group of a composite knot is given as an amalgamated product of the knot groups of the summands. [5]

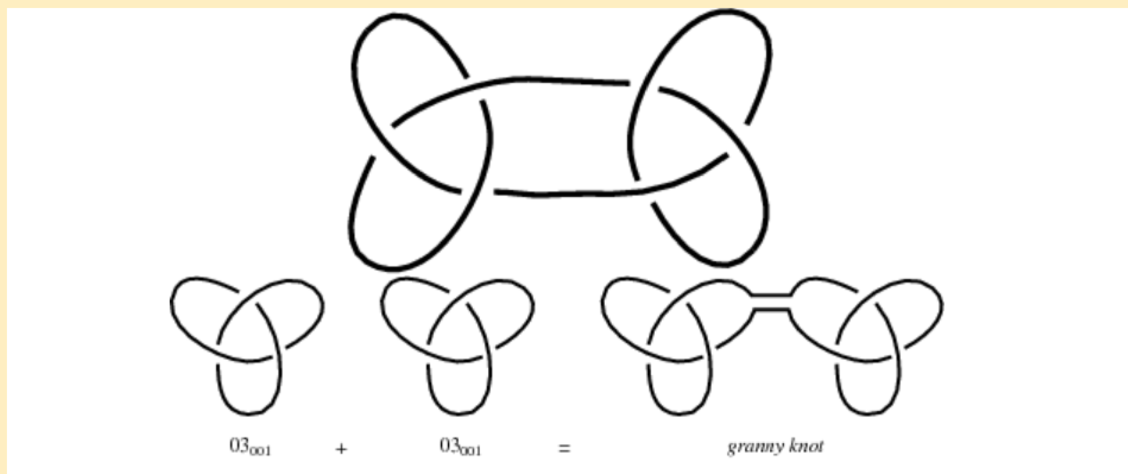
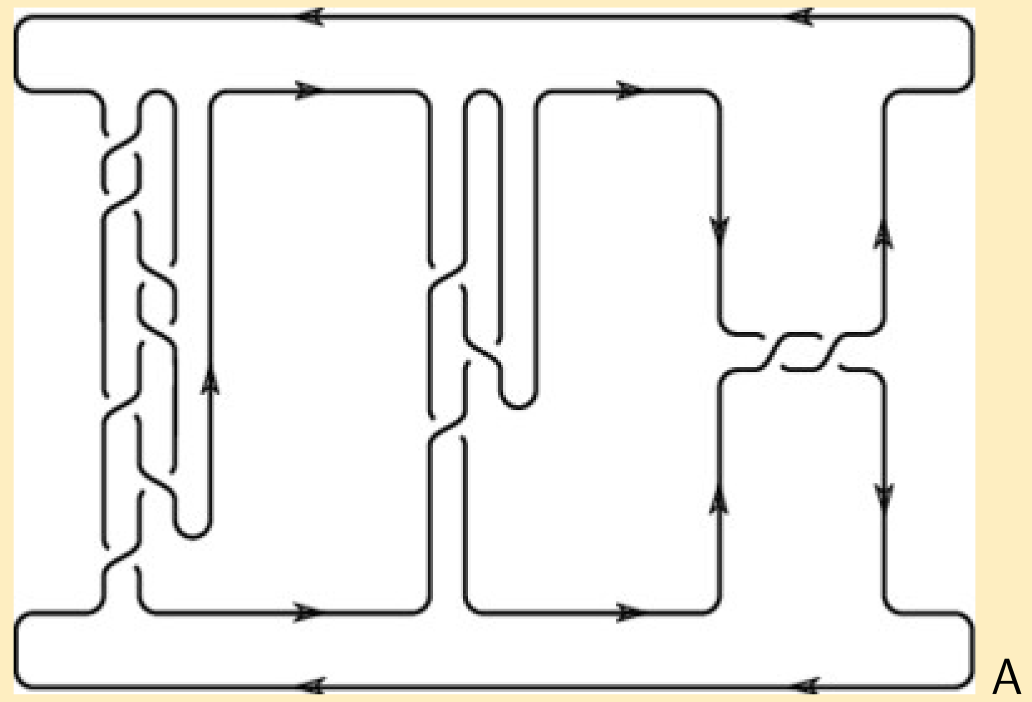


Figure: construction of a composite knot known as a granny knot[6]

## Results

We currently have calculated the dimensions of 20 hyperbolic knots from the library, up to a hyperbolic volume of 10.79, and several satellite knots. The hyperbolic knots for which we have calculated the dimension have all resulted in $\dim(K) = 1$. The satellite knots have given mixed results, with some having $\dim(K) = 1$ but many have dimension greater than 1.
We found theoretical results related to composite knots and Montesinos knots.
A hyperbolic knot with a dimension $n$ could be combined with another knot by connected sum; the resulting composite knot must have a dimension of at least $n$.




Montesinos Knot [2]

## Potential Future Work

Potential future work would include refining and automating calculations and computations, investigating current and additional conjectures, exploring other angles to approach the problems presented, and investigating possible bounds on the dimensions of classes of knots.

## References

[1] *The Knot Book An Elementary Introduction to the Mathematical Theory of Knots.*

[2] Yuanan Diao, Claus Ernst, Gabor Hetyei, and Pengyu Liu. A diagrammatic approach for determining the braid index of alternating links. 01 2019.

[3] Robert W. Ghrist, Philip J. Holmes, and Michael Sullivan. *Knots and links in three - dimensional flows.* Springer, 1997.

[4] Luisa Paoluzzi and Joan Porti. Non-standard components of the character variety for a family of montesinos knots. *Proceedings of the London Mathematical Society*, 107(3):655–679, 2013.

[5] Saif Sultan. `https://web.northeastern.edu/beasley/MATH7375/Lecture7.pdf`, 2016.

[6] Eric W. Weisstein. Granny knot. `https://mathworld.wolfram.com/GrannyKnot.html`.