Statistics in Deformations of Large Knots

George Andrews, Savannah Crawford, Mathew Hasty
Dr. Sean Lawton

George Mason University, MEGL

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Introduction

- Goal: to discover trends in knots based on a measure of complexity, the Krull dimensions of the character variety of the knot group.
- We need only observe "large" knots, as all "small" knots have dimensions 1 [4].
- We calculate dimension from a knot K by first considering its complement. We then obtain/calculate the fundamental group. Next, we calculate the character variety. Finally, we calculate the Krull dimension.

$$K \to K^C \to \pi_1(K^C) \to \mathcal{X}(\pi_1(K^C)) \to \dim(\mathcal{X}(\pi_1(K^C)))$$

Definitions

- Knot: An embedding of a circle, S^1, into $\mathbb{R}^3$.
- Large Knot: A knot whose complement admits a closed essential surface.
- Closed Essential Surface: An incompressible surface with no boundary that is homotopically nontrivial. Equivalently, its inclusion induces an injection on fundamental groups.
- Fundamental Group: The set of equivalency classes of based loops together with the operation of path-product. Here equivalence is given by homotopy.
- Character Variety: The set of equivalence classes of group homomorphisms from the fundamental group to $SL_2(\mathbb{C})$, where equivalence is given by conjugation in $SL_2(\mathbb{C})$.
- Krull Dimension: A measure of the length of prime ideals in a ring. Equivalently, for our project, a measure of the degrees of freedom there are with respect to the choices of homomorphisms in the character variety.

Regina

Software for 3-manifold topology
and normal surface theory

- Using *Regina*[3], a topological computation software, we gathered fundamental group data from all prime knots with up to 12 crossings (2977 knots).

- Separated the large knots (1019) from the small knots.

- Extracted the necessary data using a *Python* organization script.

Character Variety and Dimension Calculations

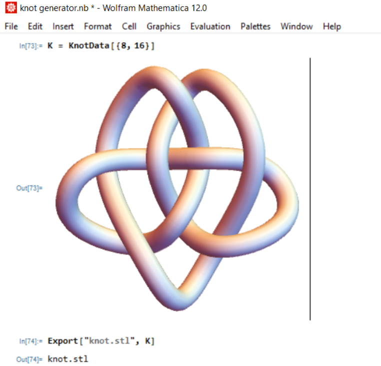
- Using code written by Ashley et al. [2] , we calculated the character varieties of each large knot's fundamental group.
- Using *Macaulay2*, we attempted to calculate the corresponding Krull dimensions.
- These calculations proved too much for our computers to handle.

Character Variety and Dimension Calculations (cont.)

- Our motivating example suggested the ordering of the knots that the *Regina* library used (crossing number) doesn't correlate to dimension or computational complexity.
- **Hyperbolic volume** is a knot invariant of hyperbolic knots. It is the volume of the knot's complement with respect to its complete hyperbolic volume.[1]
- The data suggests a correlation with hyperbolic volume of the knot compliment.
- We reordered the knots based on hyperbolic volume.
- We then calculated the dimension of 20 knots, up to hyperbolic volume 10.79

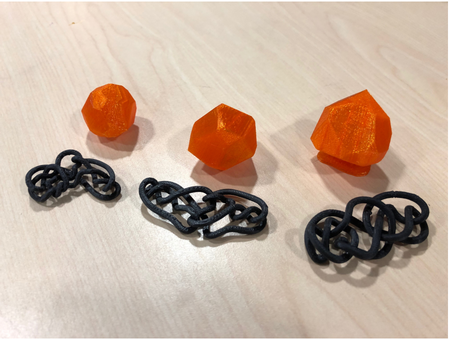
Mathematica KnotData

- Mathematica's built-in data library KnotData[5]
- Simple to run and utilize
- Has information on all prime knots with up to 10 crossings, but of the large knots, only has visualizations for 8_{16} and 8_{17}



Using SeifertView

- SeifertView has visualizations of all prime knots with up to 10 crossings[6]
- Can generate knots and Seifert surfaces
- Has more room for customization



Results

- All dimensions calculated so far have been 1.
- Conjecture: There exist "volume thresholds" which correlate to the dimensions; that is, all knots result in dimension 1 up to a certain hyperbolic volume of the complement, and similarly for other dimensions.
- Created a number of models to visualize the knots and associated data.

Future Work

- Automate calculations.
- Investigate current and further conjectures based on current and future data.
- Find a knot with greater dimension than 1.

References

[1] C. Adams, M. Hildebrand, and J. Weeks. Hyperbolic invariants of knots and links. *Transactions of the American Mathematical Society*, 326(1):1–56, 1991.

[2] C. Ashley, J. Burelle, and S. Lawton. Rank 1 character varieties of finitely presented groups. *Geometriae Dedicata*, 192(1):1–19, Sep 2017.

[3] B. A. Burton, A. Coward, and S. Tillmann. Computing closed essential surfaces in knot complements, 2012.

[4] D. Cooper, M. Culler, H. Gillet, D. D. Long, and P. B. Shalen. Plane curves associated to character varieties of 3-manifolds. *Inventiones mathematicae*, 118(1):47–84, Dec 1994.

[5] W. R. Inc. Mathematica, Version 12.0.

[6] J. V. Wijk and A. Cohen. Visualization of seifert surfaces. *IEEE Transactions on Visualization and Computer Graphics*, 12(4):485–496, 2006.