Statistics in Deformations of Large Knots

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Introduction

The goal of this project is to discover trends in knots based on a measure of "complexity." Here complexity is measured by a specific knot invariant: the Krull dimension of the character variety of the knot group. For our project, we need only look at "large" knots as all "small" knots have dimension 1.[5] Given a knot K, we look at its complement in 3-space, K^C. Then, we take the fundamental group of this space $\pi_1(K^C)$ and calculate the character variety, $\mathcal{X}(\pi_1(K^C))$. Finally, we calculate the Krull dimension of the character variety. This dimension is a knot invariant; that is, if two knots yield different dimensions, they are fundamentally different.

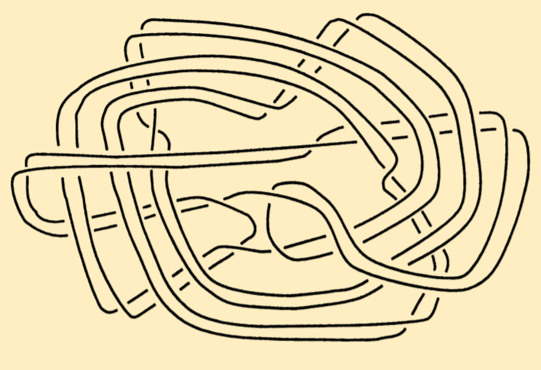


Figure: Census Knot 343, first example of large knot with dimension 1

Definitions

A **Knot** is an embedding of a circle (S^1) into $\mathbb{R}^3$ or S^3.

A **Large Knot** is knot whose complement admits a closed essential surface.

A **Closed Essential Surface** is an unbounded surface (in that it has no edges) which is nontrivial in terms of homotopy.

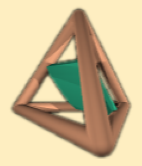
Fundamental Group Given a space X with a basepoint x_0, the **fundamental group** of (X, x_0) is given by the set of classes of loops based at x_0. The equivalence defining the classes is given by homotopy; that is, two loops are equivalent if one can be continuously deformed into the other. The operation on this set is path-product, or concatenation of paths.

The **Character Variety** of a group is the set of equivalence classes of group homomorphisms.[2] For our project, equivalence is given by conjugation with $SL_2(\mathbb{C})$.

The Krull **Dimension** of a ring is a measure of the length of prime ideals.

Gathering Fundamental Groups of Large Knots

Using *Regina*, a software developed for topological computation, we gathered the fundamental group representations of 2977 non-trivial prime knots with number of crossings ≤ 12.[4] Burton, et al. determined which of these knots were large or small. We extracted the data pertaining to just the large knots (1019 knots) using a basic search script in *Python* along with *Regina*'s built in functionality.



Calculating Character Varieties

Using code written by Ashley et al.[2], we calculated the character varieties of each large knot's fundamental group.

Calculating Dimension (Attempt 1)

Using *Macaulay2*, an algebra software, we attempted to calculate the resulting dimensions from the character varieties. However, the complexity of the computations proved too much for the computers to handle. Even with memory allotment maxed out to 500GB, there was not enough RAM to complete the computations.

Calculating by Hyperbolic Volume

The **hyperbolic knot** is a knot that can be given a metric of constant curvature -1. Thus far, all of the large knots we've found are *hyperbolic*. These knots have a knot invariant: **hyperbolic volume**, which is the volume of the knot's complement with respect to its complete hyperbolic volume.[1]

Since our motivating example has low hyperbolic volume, 6.259, we sorted the large knots by hyperbolic volume as opposed to number of crossings. For further chances of success, we also moved our dimension calculations to the more robust Magma. [3] This has led to a conjecture that there exist "volume thresholds" which correlate with resulting dimensions. Thus far, we have sorted our character varieties by volume, and the 20 large knots with least hyperbolic have dimension 1.

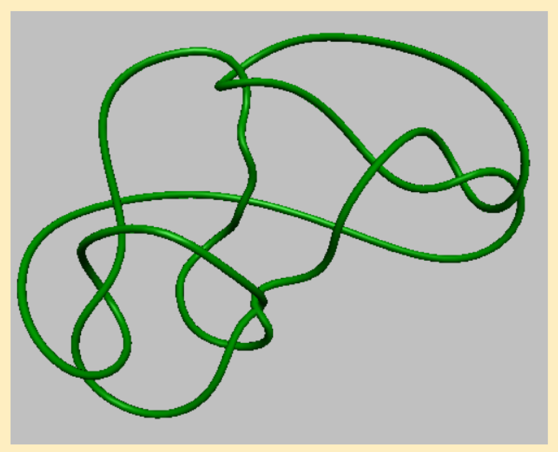


Figure: Knot 10_{153}: Large knot with the smallest hyperbolic volume with up to 12 crossings.

Visualization

In tandem with our dimension calculations, we have worked to visualize these knots. The Mathematica data library, KnotData, has information on all prime knots up to 10, but it only has 3D visualizations for two of the large knots in our data set: 8_{16} and 8_{17}. [6]

Figure: 8_{17}, Second large knot in Rolfsen knot table when sorted by crossing number, Volume:10.99, Dimension: currently incalculable

So far we've also modeled knots with SeifertView, which has a built-in library of all prime knots up to 10 crossing and gives the user flexibility to customize the visualization. In addition to generating visualizations of knots, SeifertView generates the Seifert surface of knots and links. The Seifert surface is like the result of dipping a knot in soap film. The result is an orientable surface bound by the knot. [7]

Figure: 10_{153}, Smallest volume knot in library, Volume: 7.37, Dimension: 1

Results

We currently have calculated the dimensions of 20 knots from the library, up to a hyperbolic volume of 10.79. So far, all of the knots have had dimension 1, which does give some amount of credence to our working conjecture regarding the existence of "volume thresholds" that correlate to dimension.

Future Work

Our future work consists of expanding our data set by calculating the dimensions of the remaining knots from the Rolfsen Table. To this end, we plan on automating the processes involved in the calculations and potentially running calculations through the Mason ARGO cluster. Further, we aim towards exploring current conjectures formed during this project, as well as developing other conjectures based on any additional data we collect. As for our conjecture regarding "volumes thresholds", we aim to eventually find a knot that has dimension 2; which would give us a potential lead on where (or if) there exists a threshold between dimensions of 1 and 2.

References

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