Statistics in Deformations of Large Knots

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May 8, 2020

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Introduction

- Goal: to discover trends in knots based on a measure of complexity, the Krull dimension of the character variety of the knot group.

- We calculate dimension from a knot K by first considering its complement, $K^C := \mathbb{R}^3 - K$. We then calculate the fundamental group. Next, we calculate the character variety. Finally, we calculate the Krull dimension.

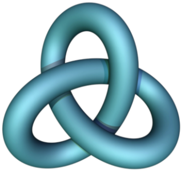
$$K \implies K^C \implies \pi_1(K^C) \implies \mathcal{X}(\pi_1(K^C)) \implies \dim(\mathcal{X}(\pi_1(K^C)))$$

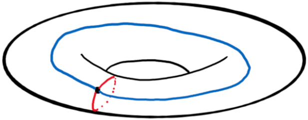
  For sake of simplicity, we will denote $\dim(K) := \dim(\mathcal{X}(\pi_1(K^C)))$.

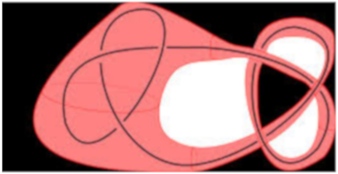
- Last semester's work has allowed us to refine our current work by classifying knots in a number of different ways; each classification has different properties that affect the calculations.

Definitions

- Knot: An embedding of a circle, S^1, into $\mathbb{R}^3$.
- Fundamental Group: The group of base loops in a space up to equivalence
- Character Variety: Set of homorphisms from the fundamental group to $\mathrm{SL}(2, \mathbb{C})$. A linear algebraic object used to describe specific types of groups.
- Krull Dimension: A measure of complexity of the ideal generated by the character variety
- Large Knot: A knot whose complement admits a closed essential surface.

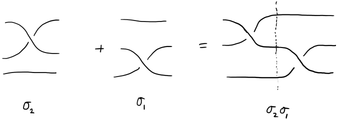






Braids

A braid is a set of parallel strands which have been interwoven. Braids can be concatenated to form new braids:

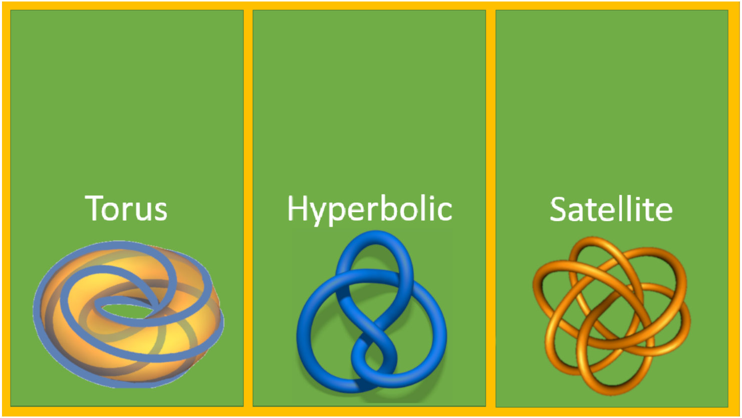


Braids are a useful way to formalize knots by the following theorem.
Alexander's Theorem: Every link (in particular every knot) is the closure of a braid.
Using Alexander's Theorem, organizing and manipulating knots is made simpler as we can instead manipulate the corresponding braid algebraically.

Outline

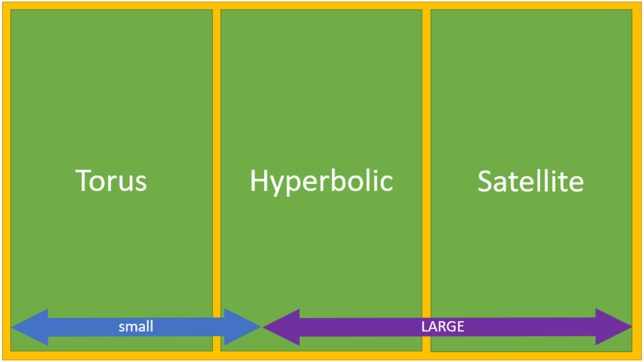
Classification of Knots

One way knots can be classified is by type. There are three types of knots: torus, hyperbolic, and satellite.[1]



Classification of Knots

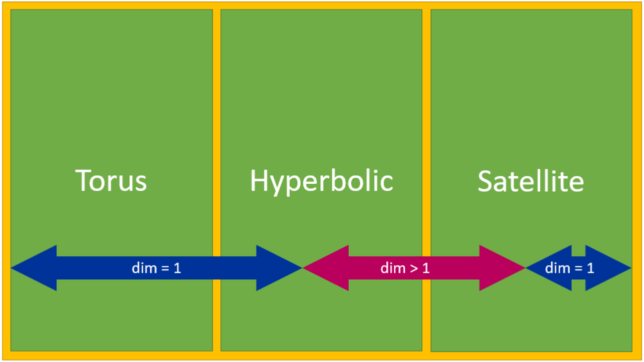
Another way knots can be classified is by them being either "large" or "small" (not large).



All torus knots are small.[4] All satellite knots are large.

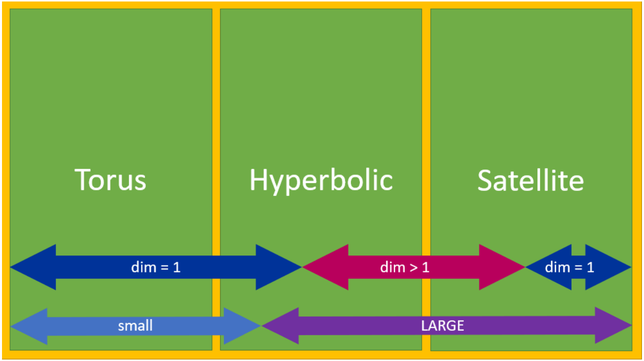
Classification of Knots

Finally, knots can be classified according to the Krull dimension, which is what our project is mainly focused on.



Classifications of Knots

These classifications overlap in a way we are currently working to understand. We have made a few observations, learned a number of facts, and discovered a few conjectures about the overlaps.



All small knots have a dimension of 1.[2]

Conjectures

- Rank Conjecture: For satellite knots, there exists $r_0 \in \mathbb{N}$ such that if $\mathrm{rank}(\pi_1(K^c)) \geq r_0$, then $\dim(K) \geq 2$.

- Volume Conjecture: For hyperbolic knots K, there exists $\alpha \in \mathbb{R}$ such that if $\mathrm{vol}(K^c) \geq \alpha$, then $\dim(K) \geq 2$ (This conjecture is stated in more detail later.)

Outline

Calculating Dimension of Large Knots

Recall

$$K \implies K^C \implies \pi_1(K^C) \implies \mathcal{X}(\pi_1(K^C)) \implies \dim(\mathcal{X}(\pi_1(K^C)))$$

- Steps $K \implies K^C \implies \pi_1(K^C) \implies \mathcal{X}(\pi_1(K^C))$ can be done for any knot in SnapPy, typically without much trouble
- $\mathcal{X}(\pi_1(K^C)) \implies \dim(\mathcal{X}(\pi_1(K^C)))$ is the problem child
- Dimension calculation in MAGMA is ideal since it utilizes a state-of-the-art algorithm for dimension calculation
- Even while utilizing the best algorithm, for nasty character varieties, this calculation becomes unreasonable for most machines
- (maybe) solution: ARGO cluster
- "Dream program": $\mathfrak{D} : \mathrm{K} \to \dim(\mathcal{X}(\pi_1(K^C)))$

Some Dimensions of Large Knots

Hyperbolic Knots: All calculated dimensions of hyperbolic knots have been 1, but there have been many hyperbolic knots whose dimension calculation was inconclusive.

Torus Knots: As previously stated, torus knots are all of dimension 1. Torus knots are completely classified.

Satellite knots: There exist a few calculated examples of satellite knots of dimensions 1 and a few examples of dimensions greater than 1.

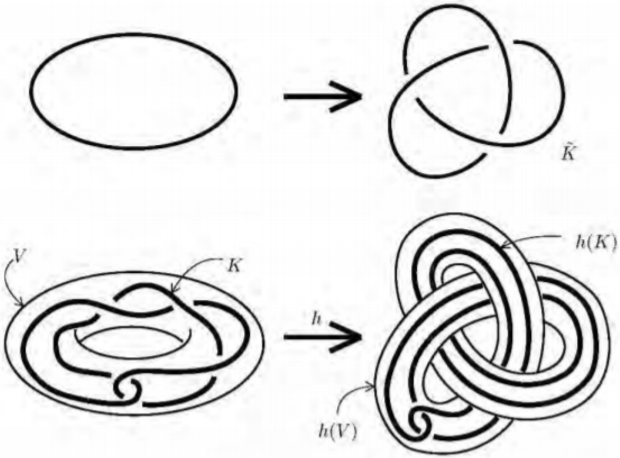
Construction of Satellite Knots

Satellite knots are formally constructed via the following method:

1. Begin with two knots, call them A and B.
2. Nontrivialy embed A into a filled torus.
3. "Tie" the filled torus into the shape of B.
4. The image of A under this "tying" is a satellite knot.

By using the braid word that describes knots (via Alexander's Theorem), this process can be automated by a computer program. This gives a method by which we may construct infinitely many nontrivial knots.

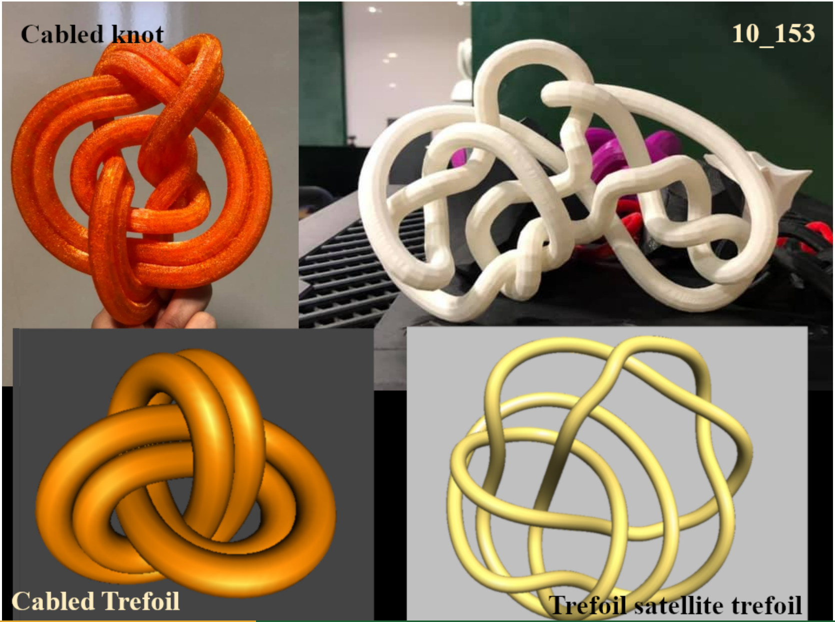
Construction of Satellite Knots



[3]

Visualizing Knots

Visualizing Knots



Outline

Weak α-Conjecture

- Def: A tangle is a complement of two disjoint strands in $\mathbb{R}^3$.
- Def: A Montesinos knot is a knot that may be viewed as a composition of rational tangles in cyclic fashion.
- Def: A Montesinos knot of Kinoshita-Terasaka type is a Montesinos knot with at least four rational tangles, one of which has a defining reduced fraction of the form $\frac{a_i}{2b_i}$. [5]

Weak α-Conjecture (contd.)

- Theorem (Weak α-Theorem): Let K be a Montesinos knot of Kinoshita-Terasaka type with $n > 3$ tangles, at least 2 of which are positive and 2 of which are negative. Denote the knot complement by M. Then the $SL_2(\mathbb{C})$-character variety $\mathcal{X}(M)$ has complex dimension at least $n - 3$. Moreover, K is a hyperbolic knot whose volume is bounded by the twist number $t(K)$:

$$\frac{v_8}{4}(t(K) - 9) \leq \mathrm{vol}(M) \leq 2v_8 t(K).$$

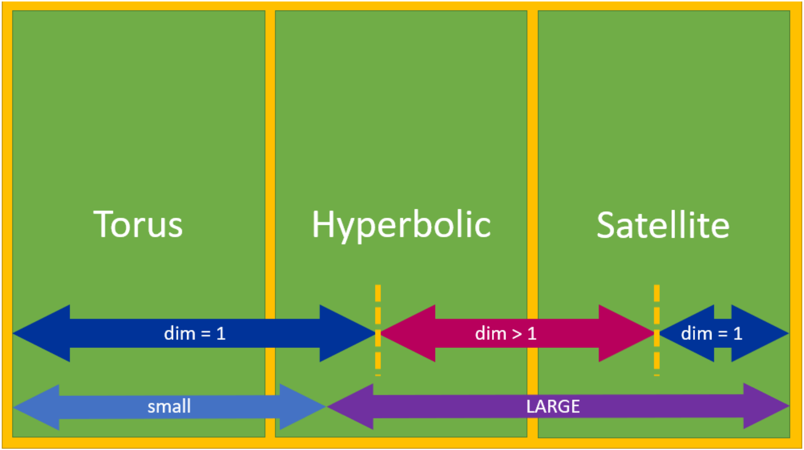
Note that $v_8 \approx 3.6638$ is the volume of the regular ideal octahedron.

- Corollary: There exists a sequence of hyperbolic knots $\{K_n\}$ such that both $\mathrm{vol}(K_n)$ and $\dim_{\mathbb{C}} \mathcal{X}(K_n)$ can be arbitrarily large.

α-Conjecture

- Conjecture (Strong α-Conjecture): Let M be the knot complement of a hyperbolic knot. For every $k > 1$ there exists $\alpha_k > 0$ such that $\text{vol}(M) \geq \alpha_k$ implies that $\dim_{\mathbb{C}} \mathcal{X}(M) \geq k$.

Recall



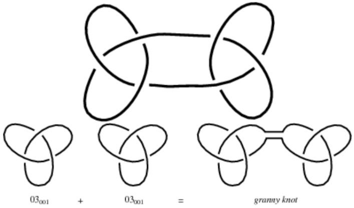
Composite Extension Problem

- Theorem: Let $K_1 \# K_2$ denote the connected sum of knots K_1 and K_2 (a composite knot made from K_1 and K_2).

$$\dim(K_1 \# K_2) \geq \max\{\dim(K_1), \dim(K_2)\}$$

- We found a way to determine lower bounds on the dimension of composite knots, which are a specific subset of satellite knots.
- This lower bound is the maximum dimension of either of the two knots which were used to construct the composite knot (either of the two summands). This result generalizes inductively to connected sums of 3 or more knots.

Illustration of the construction of a composite knot.[7] This one is denoted $3_1 \# 3_1$:



Composite Extension Problem

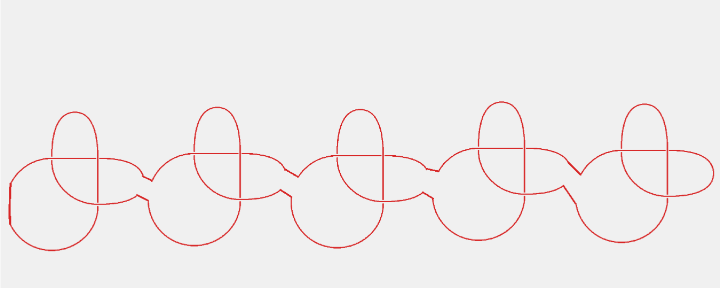
- Suppose the dimension of K_1 is greater than or equal to the dimension of K_2.
- There is a subgroup of the knot group of $K_1 \# K_2$ which is isomorphic to the knot group of K_1.[6]
- Let $i : \pi_1(K_1^C) \to \pi_1((K_1 \# K_2)^C)$ be the inclusion homomorphism. Then define $I : Hom(\pi_1((K_1 \# K_2)^C), SL_2\mathbb{C}) \to \mathrm{Hom}(\pi_1(K_1^C), SL_2\mathbb{C})$ by $I(f) = f \circ i$. The "composite extension problem" being solvable is logically equivalent to the function I being surjective (onto $Hom(\pi_1(K_1^C), SL_2\mathbb{C})$).
- If I is surjective, then the dimension of the character variety of $\pi_1((K_1 \# K_2)^C)$ is greater than or equal to the dimension of the character variety of $\pi_1(K_1^C)$

Composite Extension Problem

- In other words, if the "composite extension problem" is always solvable, then the dimension of a composite knot is at least as large as the dimension of either of the summand knots.
- Given any homomorphism $\rho : \pi_1(K_1^C) \to SL_2\mathbb{C}$, we can find a corresponding homomorphism $\widetilde{\rho} : \pi_1((K_1 \# K_2)^C) \to SL_2\mathbb{C}$ such that $\widetilde{\rho}$ restricted to the subgroup $\pi_1(K_1^C)$ is equal to ρ.
- This is possible because the knot group of a composite knot is given as an amalgamated product of the knot groups of the summand knots.[6]

Composite Extension Problem

- This result implies (through induction) that if we "glue" several knots to an initial high-dimension knot, we always have a lower bound.
- For example, the dimension of the granny knot (made with 2 trefoils) is 2. We can add several more knots (for example, 3 more trefoils) by connected sums. The resulting knot has a dimension of at least 2.



Outline

Potential Future Work

- Refine and automate calculations.
- Investigate current and further conjectures based on current and future data.
- Explore algebraic/geometric/topological approaches to calculations.
- Investigate upper and lower bounds of dimension for classes of knots

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[2] C. Ashley, J. Burelle, and S. Lawton. Rank 1 character varieties of finitely presented groups. *Geometriae Dedicata*, 192(1):1–19, Sep 2017.

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[5] L. Paoluzzi and J. Porti. Non-standard components of the character variety for a family of montesinos knots. *Proceedings of the London Mathematical Society*, 107(3):655–679, 2013.

[6] S. Sultan. `https://web.northeastern.edu/beasley/MATH7375/Lecture7.pdf`, 2016.

[7] E. W. Weisstein. Granny knot. `https://mathworld.wolfram.com/GrannyKnot.html`.