Visualizing 2x2 Matrices Using Hyperbolic Soccer

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2x2 Matrices

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- We can also consider $SL_2\mathbb{R}$, now viewed as a subgroup of $SL_2\mathbb{C}$, acting on complex 2-space $\mathbb{C}^2$.
- This isn't very easy to visualize, so instead we will shift our attention to the space $\mathbb{C}P^1$, which only has 1 complex dimension.

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- Upon restricting to the upper half-space (which we will call $\mathbb{H}^2$) we get our final product, the group $SL_2(\mathbb{R})$ acting on $\mathbb{H}^2$.

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- Another subgroup, which is quite important geometrically, is the subgroup $K = \{ \begin{bmatrix} t & 0 \\ 0 & \frac{1}{t} \end{bmatrix} \mid t > 0 \}$

$\mathrm{SL}_2(\mathbb{R})$ and Hyperbolic Space

- Note that there is an element of $\mathrm{SL}_2(\mathbb{R})$ that doesn't "do" anything in $\mathbb{C}\mathrm{P}^1$. If we consider this the "same" as the identity, and take every element of $\mathrm{SL}_2(\mathbb{R})$ "up to" this element, we get a group called $\mathrm{PSL}_2(\mathbb{R})$.
- It turns out that this group is the geometry preserving group of 2-dimensional hyperbolic space (which as a set is precisely the $\mathbb{H}^2$ we've been looking at), in that it preserves hyperbolic distances.

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- It turns out that this group is the geometry preserving group of 2-dimensional hyperbolic space (which as a set is precisely the $\mathbb{H}^2$ we've been looking at), in that it preserves hyperbolic distances.
- In looking at the group $SL_2\mathbb{R}$ acting on $\mathbb{H}^2$, we are naturally drawn to the subject of Hyperbolic Geometry.

$\mathrm{SL}_2\mathbb{Z}$

- Looking back to the cyclic subgroups discussed before, we can combine them to get a group $\mathrm{SL}_2\mathbb{Z}$. Geometrically, this group describes a certain tiling of hyperbolic space.
- Viewing hyperbolic space with this tiling, we can restrict our view to one tile, as the others can be retrieved by some element of $\mathrm{SL}_2\mathbb{Z}$.

$\mathbb{H}^2/\mathbf{SL}_2\mathbb{Z}$

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- Geodesics in $\mathbb{H}^2$ translate to geodesics in this new surface,