Content Formulas in Algebras Derived from Grassmanians

Shrunal Pothagoni, Jae-Moon Hwang, Aleyah Dawkins

Under the supervision of Dr. Neil Epstein
George Mason University, MEGL

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Table of Contents

Central Question

- The central question we want to answer is whether or not the Dedekind-Mertens property holds for a commutative ring with a Grassmannian extension?

Some Definitions

- All rings are assumed to be commutative, unital, Noetherian. We will use R to denote a ring.
- Ring - An abelian group that has a second operation that is distributive, associative, and has an associated identity element.
- Ideal - A subgroup of a ring where if $r \in R$ and $x \in I$, then rx are elements of I.
- R-Module (M) - An abelian group with an operation $\cdot : R \times M \to M$. (i.e. A generalization of a vector space.)
- R-Algebra - An R-module with its own associative R-bilinear binary operation with a multiplicative identity.

What is a content?

- Let R be a commutative ring and x be an indeterminate over R. Then given an arbitrary polynomial f in $R[x]$, the **content** of f, is defined as the ideal $c(f)$ of R generated by the coefficients of f.
- More generally, let S be an R algebra that is free as an R module on a basis $\{e_\alpha\}_{\alpha\in\Lambda}$ and let f $= \sum_{i=1}^{n} r_i e_{\alpha_i} \in$ S, then the content is the ideal generated by $(r_1, r_2, ..., r_n)$ in R.

Some history

- Gauss showed that if f, g are two polynomials in $\mathbb{Z}[x]$, then $c(fg) = c(f)c(g)$.
- Gauss' Lemma generally does not hold unless R very closely resembles the integers or a field.
- But, Dedekind and Mertens showed in 1893 that, for any $f, g \in R[x]$ and for all R, there exists a natural number n such that $c(f)^n c(g) = c(f)^{n-1} c(fg)$.
- In other words, Dedekind-Mertens property is generalized for a polynomial extension of any commutative ring.

R-Algebras

- An R-algebra, $R \to S$, is defined to be a *weak content algebra* if for any $f, g \in S$, one has $c(f)c(g) \subseteq \sqrt{c(fg)}$.
- An R-algebra, $R \to S$, is defined to be a *content algebra* if the Dedekind-Mertens holds to be true.

More background

- Northcott (1959) wrote a paper that generalized a theorem on the contents of polynomials which includes a framework on approaching content formulas.
- Huneke and Heinzer (1998) also discovered another way to find n in regards to the Dedekind-Mertens property.
- Epstein and Shapiro (2014) also extended the notion of a content formula to power series rings, showing that $R \to R[[x]]$ is a content algebra whenever R is Noetherian.

Grassmannian Extension

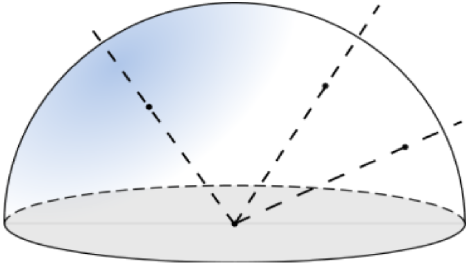
- We define the **Grassmannian**, $Gr(k, n)$, as the space which parameterizes all k-dimensional linear subspaces of an n-dimensional vector space.

- The first unknown example we are looking at is $R[Gr(2, 4)]$.

- In our research, we want to create a similar function from the unknown case $R[a, b, c, d, e, f]/\langle ab - cd + ef \rangle$ to ideals of R and see if the Dedekind-Mertens property still holds.

- We use $R[a, b, c, d, e, f]/\langle ab - cd + ef \rangle$ as it is isomorphic to $R[Gr(2, 4)]$.

Visualizing a Grassmannian



- The picture above shows a representation of $Gr(1, 3)$.
- In this case, we can represent $Gr(1, 3)$ as lines represented by the origin and a point on the unit sphere going through the origin.

Expanding on the initial question

- So, the goal of our project is to extend the notion of a content formula with a Grassmannian extension in mind.
- An answer in the negative, that is the Dedekind-Mertens property does not hold, would answer a 40 year old question.
- An answer in the positive, that is the Dedekind-Mertens property does hold, would expand the theory on Grassmannians.

Initial Approach

- We first examined an initial experiment done by Professor Epstein. The initial experiment was done in Macaulay2 and was set up to look at contents in a general way. However, the experiment would end up stalling when $n = 4$ with no answer.
- We were able to isolate why it was stalling, which was due to the Gröbner basis calculations.
- Due to the nature of our input, we suspect that our input led to some exponential calculation while it is calculating the Gröbner basis for our ideals.
- So, our next approach was to look for more efficient ways of calculating Gröbner basis.

More on the initial experiment

- Let k denote $\mathbb{Z}/101$
- Let A denote the polynomial ring, $k[w_1..w_6, x_1..x_{20}, y_1..y_6, z_1..z_{20}]$
- Let B denote the quotient ring, $A[a, b, c, d, e, f]/(a * b - c * d + e * f)$
- The initial experiment found that the Dedekind-Mertens property holds in the linear case (i.e. no power greater than 1 for a through f).
- However, we run into computational problems when we look at the quadratic case (i.e. all distinct generators of $(a, b, c, d, e, f)^2$ and from the linear case).
- We have 52 variables as they are representing arbitrary coefficients for elements in B. There are about 26 distinct combinations of a through f (as our indeterminants), and we need 2 elements from B in order to experiment with the Dedekin-Mertens property.
- We then create $f, g \in B$ and compute $c(f)^n c(g) = c(f)^{n-1} c(fg)$ where n starts at 1 until the equality holds (if it does).

Randomized Experiment

- Due to our computational difficulties we decided to change our approach.
- Since the main problem was that our polynomial base ring A is too large, we decided to assign each distinct generator an randomly generated coefficient from a smaller base ring as opposed to an arbitrary coefficient from a larger base ring.
- Not only did this method fix the computational problems, it also showed us that the Dedekind-Mertens property held true under the condition that $A = k[x, y, z]$ where the coefficients of our polynomials where randomly generated.

Gröbner Basis Algorithms

- **Buchberger's Algorithm**, created by Bruno Buchberger in 1965, is the most known algorithm for calculating Gröbner Basis.
- It is implemented in many computer algebra systems such as Macaulay2.
- Unfortunately, it is generally inefficient. If the input is not simple or "nice", then it can easily result in exponential calculations.
- However, the **F4 algorithm** (2002) was created in order to calculate Gröbner basis faster.
- There are limited implementations, but there are versions of the algorithm implemented in Magma and Faugere's own implementation.

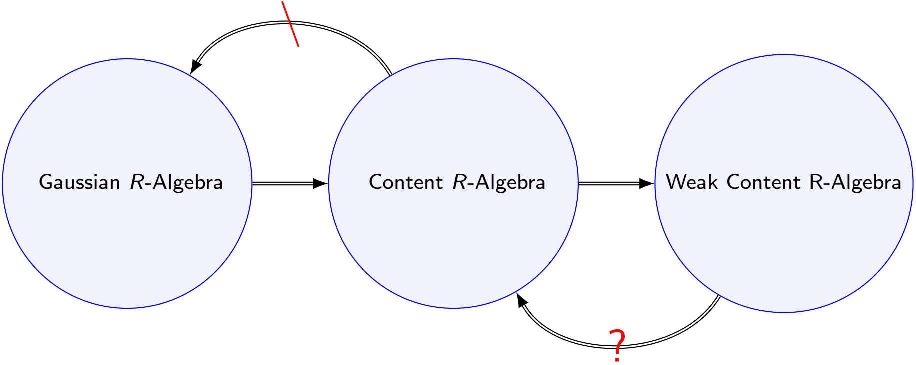
Other Software

- With the F4 algorithm in mind, we looked at various other computer algebra systems in order to import our initial approach from Macaulay2.
- We initially started with Magma, but it would ultimately fail as ideal multiplication ended up being the bottleneck rather than the Grobner basis calculation.
- We also looked at SymPy, another computer algebra system in the form of a python package. However, it seems to only have a simple implementation of a Gröbner basis algorithm and is lacking in general support.

Other Software (cont.)

- We then looked at **SageMath**, which is a computer algebra system based around Python and R.
- While it seemed to have multiple implementations to calculate a Gröbner basis, it does not support operations done on our quotient ring.
- We also looked at **CoCoA**, but we encountered the same problem as we did with SageMath.
- While there are other computer algebra systems, our problem may require novel implementations rather than relying on current existing computer algebra system if we hope to approach from a general point of view.

R-Algebras Diagram



Content Algebra $\not\Rightarrow$ Gaussian

Let $R = k[x, y]$, k be any ring

Let $S = R[t]$

Consider $g = xt - y$, $h = xt + y$, so we have $g, h \in S$

Notice $c(g) = (x, -y) = (x, y) = c(h)$

So $c(g)c(h) = (x, y)^2 = (x^2, xy, y^2)$ and $c(gh) = c(x^2t^2 - y^2) = (x^2, y^2)$

Since $xy \notin (x^2, y^2) = c(gh) \Rightarrow c(g)c(h) \neq c(gh) \Rightarrow S$ is not a Gaussian R-algebra

<u>But</u> $(x^2, xy, y^2)(x, y) = (x^3, x^2y, xy^2, y^3) = (x, y)(x^2, y^2)$

$\Rightarrow c(g)^2c(h) = c(g)c(gh)$, so the Dedekind-Mertens Lemma holds

In fact, the Dedekind-Mertens Lemma holds $\forall\ g, h \in S$.

Therefore, S is a content algebra.

Weak Content Algebra $\Rightarrow$ Content Algebra ?

- For Noetherian ring R, $R[[x_1, ..., x_n]]$ is a flat weak content algebra that was thought to not be a content algebra until 2014, as claimed by Rush in 1978.
- $R[Gr(2,4)] \cong {}^{R[a,b,c,d,e,f]}\!\big/_{\langle ab - cd + ef \rangle}$ is a weak content algebra
- If $R[Gr(2,4)]$ is not a content algebra, then this would be the first weak content algebra that is free as an R-module that is not a content algebra.

Proof Analysis

- We also tried to do proof analysis on Northcott's paper to see if we could approach our problem in a similar way.
- (Main Theorem) Let (u_σ) be an (M, G)-set with content U, (a_σ) an (R, G)-set with content A and let their G-product (v_σ) have content V. If now G has no non-zero elements of finite order, then $UA^{k+1} = VA^k$.
- Northcott showed extensions like $R[t^2, t^5]$ are Content R-Algebras
- G has no non-zero elements of finite order $\Leftrightarrow$ G is an ordered semi-group $\Leftrightarrow$ G is a cancellative torsion-free semi-group
- However, the Grassmannian extension fails to be such a semi-group extension

Future Work

- Ultimately, we have a few directions that we can look towards.
- One direction would be to keep trying to modify our problem in a more specific manner and thus requiring less computations than in the general case. We have had some success in this case, and this path may be needed to gain more insight on our problem.
- Another direction is to implement our own algorithms in order to solve our problems.
- To be more specific, we can use Macaulay2 for most of more basic calculations (e.g. constructing our rings and notation) and implement our own algorithms where Macaulay2 is not as efficient (e.g. calculating the Gröbner basis).

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