Content Formulas in Algebras Derived from Grassmanians

Neil Epstein (Faculty Advisor), Shrunal Pothagoni, Jae-Moon Hwang, Aleyah Dawkins

Mason Experimental Geometry Lab

George Mason University

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Central Question

The central question we want to answer is whether or not the Dedekind-Mertens property holds for a commutative ring with a Grassmannian extension?

Preliminary Definitions/Concepts

- All rings are assumed to be commutative, unital, Noetherian. We will use R to denote a ring.
- Ring - An abelian group that has a second operation that is distributive, associative, and has an associated identity element.
- Ideal - A subgroup of a ring where if $r \in R$ and $x \in I$, then rx are elements of I.
- R-Module (M) - An abelian group with an operation $\cdot : R \times M \to M$. (i.e. A generalization of a vector space.)
- R-Algebra - An R-module with its own associative R-bilinear binary operation with a multiplicative identity.

More Definitions

Definition (Content)

- Let R be a commutative ring and x be an indeterminate over R. Then given an arbitrary polynomial f in $R[x]$, the **content** of f, is defined as the ideal $c(f)$ of R generated by the coefficients of f.
- More generally, let S be an R algebra that is free as an R module on a basis $\{e_\alpha\}_{\alpha \in \Lambda}$ and let $f = \sum_{i=1}^{n} r_i e_{\alpha_i} \in S$, then the content is the ideal generated by $(r_1, r_2, ..., r_n)$ in R.

Definition (Content Algebras)

- An R-algebra, $R \to S$, is defined to be a *weak content algebra* if for any $f, g \in S$, one has $c(f)c(g) \subseteq \sqrt{c(fg)}$.
- An R-algebra, $R \to S$, is defined to be a *content algebra* if the Dedekind-Mertens holds to be true.

Some Background History

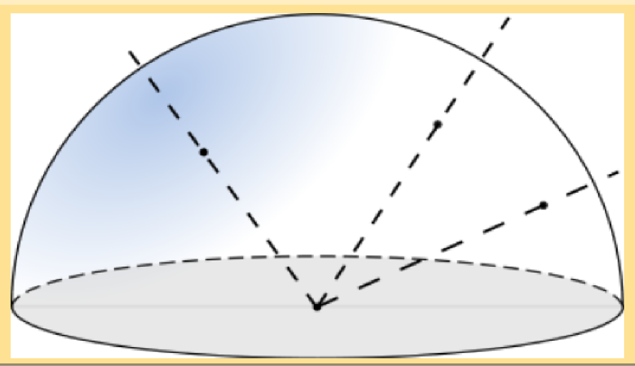
- Gauss showed that if f, g are two polynomials in $\mathbb{Z}[x]$, then $c(fg) = c(f)c(g)$.
- Gauss' Lemma generally does not hold unless R very closely resembles the integers or a field.
- But, Dedekind and Mertens showed in 1893 that, for any $f, g \in R[x]$ and for all R, there exists a natural number n such that $c(f)^n c(g) = c(f)^{n-1} c(fg)$.
- In other words, Dedekind-Mertens property is generalized for a polynomial extension of any commutative ring.

Some more recent history

- Northcott (1959) wrote a paper that generalized a theorem on the contents of polynomials which includes a framework on approaching content formulas.
- Huneke and Heinzer (1998) also discovered another way to find n in regards to the Dedekind-Mertens property.
- Epstein and Shapiro (2014) also extended the notion of a content formula to power series rings, showing that $R \to R[[x]]$ is a content algebra whenever R is Noetherian [Theorem 2.6 pg 5].

Grassmannian Extension

- We define the **Grassmannian**, $Gr(k, n)$, as the space which parameterizes all k-dimensional linear subspaces of an n-dimensional vector space.
- The first unknown example we are looking at is $R[Gr(2, 4)]$.
- In our research, we want to create a similar function from $R[Gr(2, 4)]$, using $R[a...f]/ab - cd + ef$ to ideals of R and see if the Dedekind-Mertens property still holds.
- We use $R[a, b, c, d, e, f]/ab - cd + ef$ as it is isomorphic to $R[Gr(2, 4)]$.



Visualizing a Grassmannian

- The picture to the left shows a representation of $Gr(1, 3)$.
- In this case, we can represent $Gr(1, 3)$ as lines represented by the origin and a point on the unit sphere.

Technical Approach

Initial Approach

- An initial experiment in Macaulay2 was done by Professor Epstein that looked at the problem in a general approach. However, it would end up stalling at n = 4 with no answer.
- Due to the nature of our input, we suspect that our input led to some exponential calculation while calculating the Gröbner basis for our ideals which led us to examine Gröbner basis algorithms.

Gröbner Basis Algorithms

- **Buchberger's Algorithm** is the most known algorithm for calculating Gröbner Basis and is implemented in many computer algebra systems such as Macaulay2.
- Unfortunately, it is generally inefficient.
- However, the **F4 algorithm** (2002) was created in order to calculate Gröbner basis faster, but there are limited implementations in other computer algebra systems that we examined.

Computer Algebra Systems (CAS)

- We examined several CAS: Macaulay2, Magma, SageMath, SymPy, and CoCoa
- Unforunately, none of these systems would help with our general approach due to various reasons.
- Magma would end up stalling in calculating the product of our ideals, while SymPy had limited documentation and little support with gröbner basis algorithms.
- SageMath and CoCoa had similar issues with constructing our Grassmannian ring.

Some Success

- While the general approach was too computationally expensive, we narrowed our scope and worked with more specfic rings such as $R = k[x, y, z]$, where k is any field.
- The Dedekind-Mertens property did hold when using this ring as our coefficients, but it should be noted that this only gives us more insight on the question rather than a complete answer.

Proof Analysis of Northcott's Paper

- We also tried to do proof analysis on Northcott's paper to see if we could approach our problem in a similar way. However, his paper does not easily extend to our situation as our Grassmannian extension fails to be a semi-group extension which Northcott uses.
- Northcott showed extensions like $R[t^2, t^5]$ are Content R-Algebras

Future Work

- Ultimately, we have a few directions that we can look towards. One direction would be to keep trying to modify our problem in a more specific manner and thus requiring less computations than in the general case. We have had some success in this case, and this path may be needed to gain more insight on our problem.
- Another direction is to implement our own algorithms in order to solve our problems. To be more specific, we can use Macaulay2 for most of more basic calculations (e.g. constructing our rings and notation) and implement our own algorithms where Macaulay2 is not as efficient (e.g. calculating the Gröbner basis).

Acknowledgments

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