Geometry of Complex Networks

Savannah Crawford, Chris Thorton
Matt Holzer

George Mason University, MEGL

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Table of Contents

Interested in Learning

- We are interested in the underlying geometry of a complex network.
- We want to associate a space to the network.
- We then want to embed our network into that space that maintains community structure and node distance.
- Possible applications:
    - Link Prediction
    - Routing

Basic Definitions

- What exactly is a Complex Network?
  - A network is graph, $G = (V, E)$ with V vertices and E edges.
  - A complex network is a graph with non-trivial features that do not occur in simple networks such as lattices or random graphs but often occur in graphs modelling real systems.
  - General features of real world networks: sparse, algebraic degree distribution (i.e. hubs exist), community structure, exponential growth of neighbors.

Why Hyperbolic Space

Consider the 6 degrees of separation.

There are 7.6 Billion people in the world.

If you were to isometrically embed the network of friendships in Euclidean space you would need to fit the nodes a Sphere with Radius ≤ 3.

If you instead put this in Hyperbolic space the distances locally are Euclidean-like, but grow exponentially as nodes get further apart.

Airline Network Information

Global airline network as of 2001 – Guimera *et. al* 2005

- 3618 cities (nodes) with $14,142$ connections (edges)
- Includes passenger and freight routes and air taxis
- Diameter is 17 representing travel from Wasu, Papa New Guinea to Brieze Norton Air Force Base
- Exhibits small world property – algebraic degree distribution with average shortest path between cities of 4.4
- More clustered than a random network due to spatial structure

Measuring Hyperbolicity (δ-Hyperbolicity)

- δ is a scalar value.
- δ tells how "tree-like" a graph is.
- The δ of a tree is always 0.
- There are several ways to calculate δ-hyperbolicity which all lead to brute force algorithms taking time proportional to n^4, where n is the number of vertices in the network.

Four Point Condition and δ

In a graph $G = (V, E)$, given four vertices x, y, u, and $v \in V$ with $d(x, y) + d(u, v) \geq d(x, u) + d(y, v) \geq d(x, v) + d(y, u)$, the hyperbolicity of the quadruple x, y, u, v denoted as $\delta(x, y, u, v)$ is defined as:

$$\delta(x, y, u, v) = \frac{d(x,y)+d(u,v)-(d(x,u)+d(y,v))}{2}$$

The δ of a graph is the largest δ for all four point combinations.

Examples

Tree

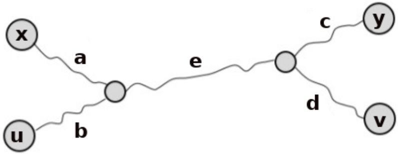

Figure: Courtesy of [2]

$$\begin{gathered} d_{xu} + d_{yv} = a + b + c + d \\ d_{xv} + d_{yu} = a + e + d + b + e + c \\ d_{xy} + d_{uv} = a + e + c + b + e + d \\ \text{so } \delta = 0 \end{gathered}$$

“Tree” with Shortcuts

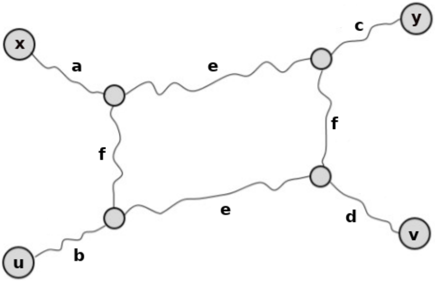

Figure: Courtesy of [2]

$$d_{xu} + d_{yv} = a + b + c + d + 2f$$
$$d_{xv} + d_{yu} = a + e + d + b + e + c + 2f$$
$$d_{xy} + d_{uv} = a + e + c + b + e + d$$
$$\text{so, } \delta = f \text{ or } \delta = e$$

Four Point Condition and δ

In a graph $G = (V, E)$, given four vertices x, y, u, and $v \in V$ with $d(x, y) + d(u, v) \geq d(x, u) + d(y, v) \geq d(x, v) + d(y, u)$, the hyperbolicity of the quadruple x, y, u, v denoted as $\delta(x, y, u, v)$ is defined as:

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The δ of a graph is the largest δ for all four point combinations.

δ-Hyperbolicty Algorithm

Algorithm 1: Hyperbolicity

Input: G is a 2-connected graph
Input: `pairs` is the list of the $\binom{n}{2}$ pairs of vertices sorted by decreasing distances.
Result: δ, the hyperbolicity of G (observe that $2\delta = \mathrm{h}_{\mathtt{diff}}$).

Let $\mathrm{h}_{\mathtt{diff}} := 0$;
for $1 \leq i < \binom{n}{2}$ **do**
    $(a, b) :=$ `pairs`$[i]$;
    for $0 \leq j < i$ **do**
        $(c, d) :=$ `pairs`$[j]$;
        $\mathrm{h}_{\mathtt{diff}} := \max\{\mathrm{h}_{\mathtt{diff}}, \delta_{\mathtt{diff}}(a, b, c, d)\}$;
        if $\mathtt{d}(a, b) \leq \mathrm{h}_{\mathtt{diff}}$ **then**
            return $\mathrm{h}_{\mathtt{diff}}/2$
return $\mathrm{h}_{\mathtt{diff}}/2$

Figure: Courtesy of [1]

Some results of our computations

Connections greater than	Number of Nodes	δ	Diameter
200	3	0	1
190	4	0	1
185	5	0	1
180	6	0.5	2
170	8	0.5	2
140	12	0.5	2
130	15	0.5	2
120	16	0.5	2
110	25	0.5	2
100	33	0.5	2
90	42	0.5	2
80	54	0.5	2
70	64	1	3
60	79	1	3
50	102	1	3
40	137	1	3
30	187	1	4
20	287	1	4
15	398	1	5
13	449	1	6
11	501	1	6
10	538	1	∞
9	594	1	6
8	638	1	6
7	692	1	7
6	781	1	7
5	915	1	∞
4	1106	1	8
3	1398	1.5	∞
2	1874	1.5	∞
1	2880	1.5	15
0	3618	1.5	17

References

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2. Alrasheed, H. M., (2018, May). δ-Hyperbolicity in Real-World Networks: Algorithmic Analysis and Implications. Retrieved on November 1, 2018 from https://etd.ohiolink.edu/ !etd.send_file?accession=kent1526411510583146disposition=inline.
3. Verbeek, K., Subhash, S. (2016, December). Metric Embedding, Hyperboolic Space, and Social Networks. Retrieved on November 1, 2018 from https://www.cs.ucsb.edu/ suri/psdir/SoCG14.pdf.

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