# Geometry of Complex Networks

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## Complex Network

A network is graph, $G = (V, E)$ with $V$ vertices and $E$ edges.

A complex network is a graph with non-trivial features that do not occur in simple networks such as lattices or random graphs but often occur in graphs modeling real systems.

General features of real world networks: sparse, algebraic degree distribution (i.e. hubs exist), community structure, exponential growth of neighbors.

## Global Airline Network as of 2001 – Guimera *et. al* 2005

- 3618 cities (nodes) with 14,142 connections (edges)
- Includes passenger and freight routes and air taxis
- Diameter is 17 representing travel from Wasu, Papa New Guinea to Brieze Norton Air Force Base
- Exhibits small world property – algebraic degree distribution with average shortest path between cities of 4.4
- More clustered than a random network due to spatial structure

## Finding a Space for Complex Networks

According to Facebook Research, on average, there are three and a half degrees of separation between Facebook users. This corresponds to an average distance of 4.5 between users, so if one were to embed the social network of Facebook into Euclidean space, the graph would need to fit into a circle of radius 3. This means one would need to fit all 1.5 billion nodes into the circle, while maintaining local distances of one between Facebook friends. This graph would need to be in a high dimension, and may not preserve distance exactly. Hyperbolic space allows for Euclidean-like distances locally, but distance grows exponentially, meaning that the graph would fit better.

## $\delta$-Hyperbolicity

### Calculating Hyperbolic-ness

- $\delta$ is a scalar value.
- $\delta$ tells how "tree-like" a graph is.
- The $\delta$ of a tree is always 0.
- There are several ways to calculate the $\delta$-hyperbolicity which all lead to brute force algorithms taking time proportional to $n^4$, where $n$ is the number of vertices in the network.
- Since trees embed isometrically into hyperbolic space, a graph with a small $\delta$ will embed well into hyperbolic space, $\mathbb{H}^n$.

## Four Point Condition

In a graph $G = (V, E)$, given four vertices $x$, $y$, $u$, and $v \in V$ with $d(x,y) + d(u,v) \geq d(x,u) + d(y,v) \geq d(x,v) + d(y,u)$, the hyperbolicity of the quadruple $x$, $y$, $u$, $v$ denoted as $\delta(x,y,u,v)$ is defined as:

$$\delta(x,y,u,v) = \frac{d(x,y)+d(u,v)-(d(x,u)+d(y,v))}{2}$$

The $\delta$ of a graph is the largest $\delta$ for all four point combinations.

## Four Point Condition Examples

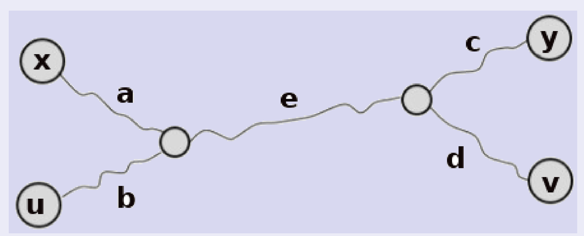
$\delta$ of points in a tree

$d_{xu} + d_{yv} = a + b + c + d$

$d_{xv} + d_{yu} = a + e + d + b + e + c$

$d_{xy} + d_{uv} = a + e + c + b + e + d$

so $\delta = 0$



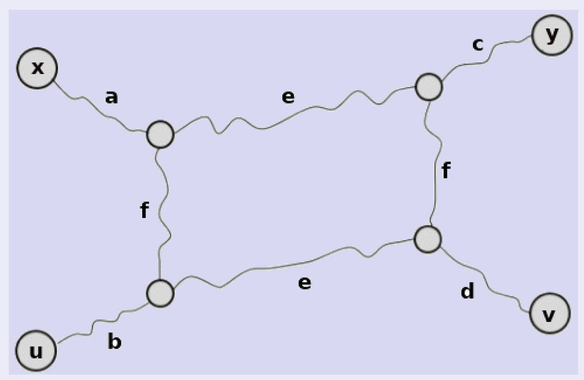

$\delta$ of a graph with shortcuts

$d_{xu} + d_{yv} = a + b + c + d + 2f$

$d_{xv} + d_{yu} = a + e + d + b + e + c + 2f$

$d_{xy} + d_{uv} = a + e + c + b + e + d$

so, $\delta = f$ or $\delta = e$




## $\delta$ Hyperbolicity of Airline Network and it's subsets

| Connections greater than | Number of Nodes | $\delta$ | Diameter |
|---|---|---|---|
| 200 | 3 | 0 | 1 |
| 190 | 4 | 0 | 1 |
| 180 | 6 | 0.5 | 2 |
| 140 | 12 | 0.5 | 2 |
| 100 | 33 | 0.5 | 2 |
| 80 | 54 | 0.5 | 2 |
| 60 | 79 | 1 | 3 |
| 20 | 287 | 1 | 4 |
| 15 | 398 | 1 | 5 |
| 10 | 538 | 1 | $\infty$ |
| 9 | 594 | 1 | 6 |
| 8 | 638 | 1 | 6 |
| 7 | 692 | 1 | 7 |
| 6 | 781 | 1 | 7 |
| 5 | 915 | 1 | $\infty$ |
| 4 | 1106 | 1 | 8 |
| 3 | 1398 | 1.5 | $\infty$ |
| 2 | 1874 | 1.5 | $\infty$ |
| 1 | 2880 | 1.5 | 15 |
| 0 | 3618 | 1.5 | 17 |

## Hyperbolicity Calculation Algorithm

**Algorithm 1:** Hyperbolicity

```
Input: G is a 2-connected graph
Input: pairs is the list of the (n choose 2) pairs of vertices sorted by decreasing distances.
Result: δ, the hyperbolicity of G (observe that 2δ = h_diff).
Let h_diff := 0;
for 1 ≤ i < (n choose 2) do
    (a, b) := pairs[i];
    for 0 ≤ j < i do
        (c, d) := pairs[j];
        h_diff := max {h_diff, δ_diff(a, b, c, d)} ;
        if d(a, b) ≤ h_diff then
            return h_diff /2
return h_diff /2
```

## Conclusions and Future Work

We cannot isometrically embed the Airline Network or it's subsets with nodes of degree $\leq 180$ into $\mathbb{R}^n$. Using a similarity embedding may be the most achievable quality embedding.

In the future, we will relax the requirement for isometric preservation of the distance between nodes wit ha similarity embedding.We would specify that connected nodes have a distance $\leq 1$ and disconnected nodes have a distance $> 1$. We are also exploring what algorithms can be used to embed this network into $\mathbb{H}^n$ since complex networks do not embed isometrically into $\mathbb{R}^n$.

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## References

1. Alrasheed, H.M., (2018, May). $\delta$-Hyperbolicity in Real-World Networks: Algorithmic Analsis and Implications. Retrieved on November 1, 2018 from https://etd.ohiolink.edu/!etd.sendfile?accession=kent1526411510583146disposition=inline.
2. Borassil, M., Coudert, D., Crescnzi, P. Marino, A. (2012, September). On computing the Hyperbolcitiy of Real-World Graphs. Retrieved on November 1, 2018 from https://arpi.unipi.it/retrieve/handle/11568/790760/191185/main.pdf.
3. Verbeek, K., Subhash, S. (2016, December). Metric Embedding, Hyperbolic Space, and Social Networks. Retrieved on November 1, 2018 from https://cs.ucsb.edu/suri/psdir/SoCG14.pdf.