Visualizing Geometric Flows: First Semester

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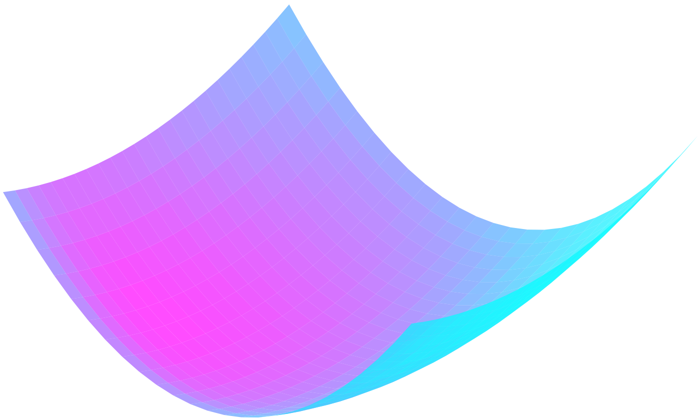
Introduction



Heat Equation in $\mathbb{R}^n$

- Heat equation in $\mathbb{R}^n$:

$$\frac{\partial u(x,t)}{\partial t} = -c^2 \sum_{i=1}^{n} \frac{\partial^2 u(x,t)}{\partial t^2}.$$

- Laplacian in $\mathbb{R}^n$ is

$$\Delta(u) = \sum_{i=1}^{n} \frac{\partial^2 u(x,t)}{\partial t^2}.$$

- Heat equation can be expressed as:

$$\frac{\partial u(x,t)}{\partial t} = -\Delta u(x,t).$$

Examples of Manifolds

- Manifold is a topological space that is “Locally Euclidean”.
- Examples:
  - $\mathbb{R}^n$
  - A sphere of any dimension
  - Smooth curves and surfaces in $\mathbb{R}^n$
- One can define derivatives of functions on a manifold.

$$C^\infty(M) = \{f : M \to \mathbb{R} : f \text{ is infinitely differentiable.}\}$$

Heat Equation on a Manifold

- Heat equation on a Manifold

$$\frac{\partial u(x,t)}{\partial t} = -c^2 \Delta u(x,t).$$

- Laplacian: $\Delta : C^\infty(M) \to C^\infty(M)$.
- Important properties:
- Δ is a linear operator acting on the infinite dimensional vector space $C^\infty(M)$.
- An eigenfunction f is a function for which $\Delta f = \lambda f$
- The eigenfunctions of Δ form a basis for $L^2(M)$ which contains $C^\infty(M)$.

Overview

- **Solutions to the Heat Equation on M can be computed by computing the eigenfunctions of Δ.** (Details Later)
- Our project:
  1. Approximate a manifold M with a finite set X of data "sampled" from a manifold M.
  2. Form a "discrete Laplacian" L which acts on functions on the finite data. The operator L provably approximates Δ as we sample more data.
  3. Compute the eigenvectors of L and compute the solutions to the heat equation on the data using the eigenvectors of L.

Approximating the Laplacian on Data

- Let $X = \{x_1, ..., x_N\}$ in $\mathbb{R}^n$. Construct a complete graph with vertex set X.
- Weight each edge $\{x_i, x_j\}$ of the graph with weight

$$d_{i,j} = e^{\frac{-||x_i - x_j||^2}{\delta^2}}$$

- δ is adjustable parameter (example).
- Let K be the matrix whose entrees are $d_{i,j}$. Let D be the diagonal matrix whose diagonal entrees are the row sums of K.
- "Discrete Laplacian" $L = D - K$.

A Closer Look at L

- If L is an $N \times N$ matrix, it is a linear map that takes a vector $\overline{f}$ of length N to a vector of length N.
- Vectors of length N thus should "approximate" functions, since Δ acts on functions.
- If f is a function on M, $\bar{f}$ is f restricted to the set of data.
- Represent $\bar{f}$ as a vector:

$$\bar{f} = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}$$

Example: S^1

- On a circle, the Laplacian is given by

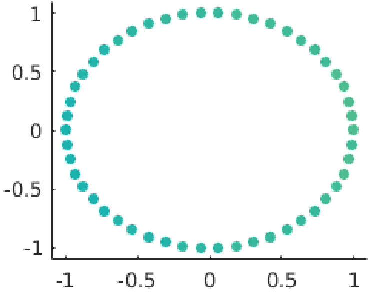
$$\Delta = \frac{\partial^2}{\partial \theta^2}$$

- The eigenfunctions of Δ are given by

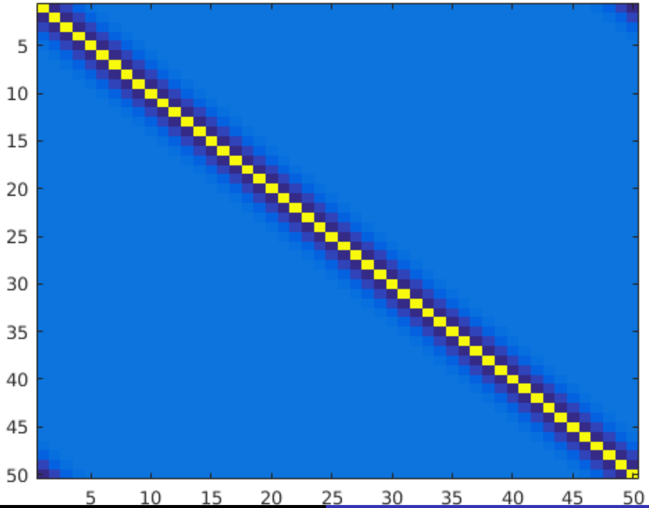
$$\cos\left(n\theta + \phi\right).$$

- Here's an example of L computing on 50 data points sampled from S^1

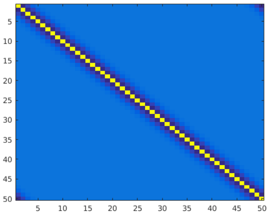
Example: 50 data points on S^1



Example: L matrix for S^1



Example S^1: Eigenvector Decomposition $L = U \Lambda U^\top$



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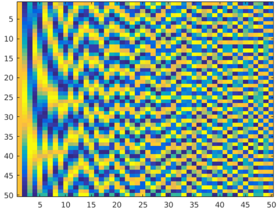
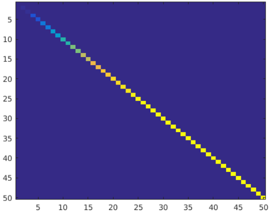
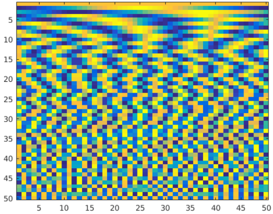


U

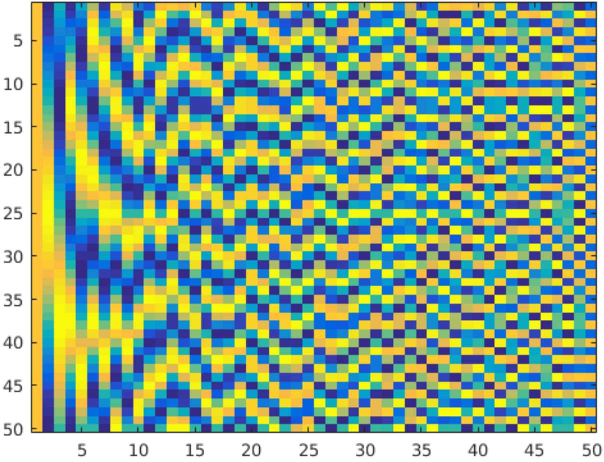


Λ

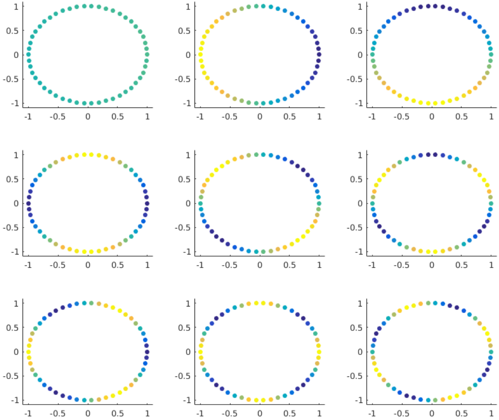


$U^\top$

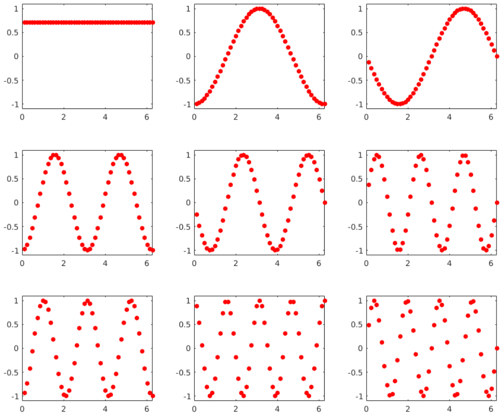
Example S^1: Matrix of Eigenvectors, U



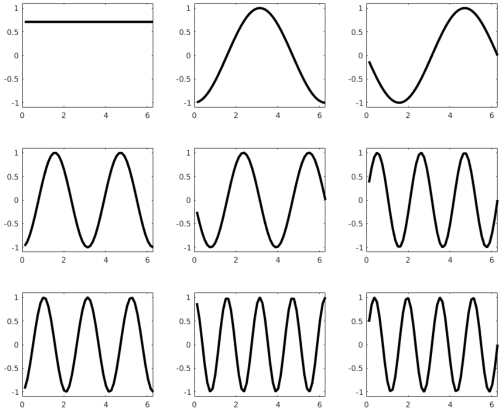
Example S^1: Eigenvectors on Data



Example S^1: Eigenvectors vs. θ



Example S^1: Connecting the dots



Solving the Classical Heat Equation

Let $u(x,t)$ be a solution to the Heat Equation. Since the eigenfunctions $\{\phi_i\}_{i=1}^{\infty}$ of Δ form a basis for $L^2(M)$, we may write:

$$u(x,t) = \sum_{i=1}^{\infty} c_i(t)\phi_i(x)$$

Plugging into the heat equation yields:

$$\frac{\partial \sum_{i=1}^{\infty} c_i(t)\phi_i(x)}{\partial t} = \Delta\left(\sum_{i=1}^{\infty} c_i(t)\phi_i(x)\right)$$

Solving the Classical Heat Equation

Partial derivatives and the Laplacian are Linear (and continuous):

$$\sum_{i=1}^{\infty} \frac{\partial c_i(t)}{\partial t} \phi_i(x) = \sum_{i=1}^{\infty} c_i(t) \Delta \phi_i(x)$$

$$\sum_{i=1}^{\infty} \frac{\partial c_i(t)}{\partial t} \phi_i(x) = \sum_{i=1}^{\infty} c_i(t) \lambda_i \phi_i(x)$$

Comparing coefficients, we get:

$$\frac{\partial c_i}{\partial t} = -c_i(t)\lambda_i$$

for all $i \in \mathbb{N}$

Solving the Classical Heat Equation

By inspection, this yields coefficients:

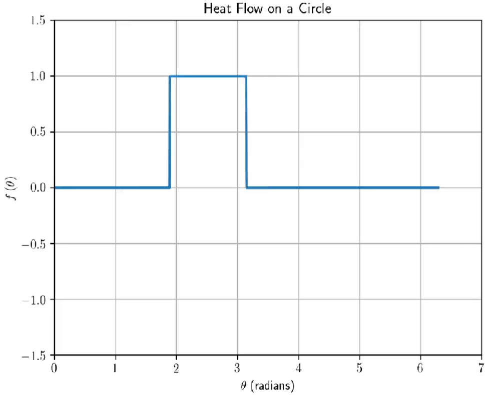
$$c_i(t) = c_i(0)e^{-\lambda_i t}$$

So solutions look like:

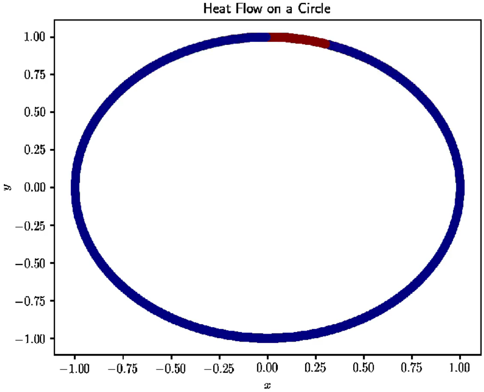
$$u(x,t) = \sum_{i=1}^{\infty} c_i(0)e^{-\lambda_i t}\phi_i(x)$$

Notice that $e^{-\lambda_i t}$ decays faster as λ_i become larger. So finding the smallest eigenvalues of Δ will yield more significant terms in the solution.

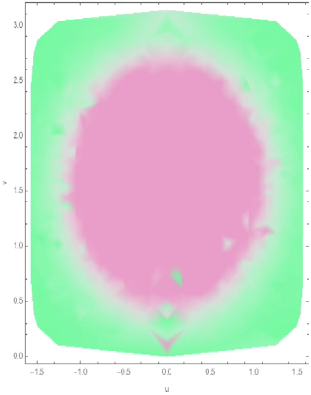
Example: Heat flow on S^1 as function of θ

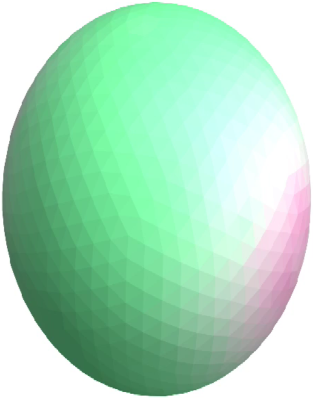


Example: Heat flow on S^1 shown on the data



Example on S^2: Intrinsic (left) Embedding (right)





Geometric Flows

- Heat Flow: Distribution of heat changes, manifold is fixed
- **Geometric Flow:** Apply the heat flow to the manifold
- As the embedding evolves, the manifold changes!
- Must recompute the Laplacian after each small time step

Geometric Flow on Ellipse