The Erdós-Szekeres Problem

The statement in 2 and 3 dimensions, and a proof technique

Tyler Russ

George Mason University

Friday, December 8, 2017

Outline of Talk

1. Introduction: Statement of the problem in 2 and 3 dimensions

2. Examples: A motivating proof in 2 dimensions and a fundamental proof in 3 dimensions

3. Case study: Outline of a partial proof for $N_3(6)$

Introduction: Statement of the problem in 2 and 3 dimensions

The Erdós-Szekeres Problem

Definition (The Erdós-Szekeres Problem in 2 dimensions)

In the plane, the number of points $N(n)$ in general position needed to guarantee a convex n-gon.

The Erdós-Szekeres Problem

Definition (The Erdós-Szekeres Problem in 2 dimensions)

In the plane, the number of points $N(n)$ in general position needed to guarantee a convex n-gon.

Definition (The Erdós-Szekeres Problem in 3 dimensions)

In 3-space, the number of points $N(n)$ in general position needed to guarantee a convex polytope (polyhedron) with n vertices.

Some definitions

Definition (General position in 2 dimensions)

In the plane, a set X of points is said to be in *general position* if no three of the points lie on a line.

Some definitions

Definition (General position in 2 dimensions)

In the plane, a set X of points is said to be in *general position* if no three of the points lie on a line.

Definition (General position in 3 dimensions)

In 3-space, a set X of points is said to be in *general position* if no three of the points lie on a line, and no four of the points lie in a plane.

Some definitions

Definition (General position in 2 dimensions)

In the plane, a set X of points is said to be in *general position* if no three of the points lie on a line.

Definition (General position in 3 dimensions)

In 3-space, a set X of points is said to be in *general position* if no three of the points lie on a line, and no four of the points lie in a plane.
(Whenever $n > 3$, the requirement that no four points be coplanar implies that no three points are collinear.)

Some more definitions

Definition (Convex hull)

The *convex hull* of a set of points X, $ConvX$, is the largest convex polytope containing all points of X.

Some more definitions

Definition (Convex hull)

The *convex hull* of a set of points X, $ConvX$, is the largest convex polytope containing all points of X.

Definition (Convexly independent in 2 dimensions)

In the plane, a set X of points is said to be *convexly independent* if

.

Some more definitions

Definition (Convex hull)

The *convex hull* of a set of points X, $ConvX$, is the largest convex polytope containing all points of X.

Definition (Convexly independent in 2 dimensions)

In the plane, a set X of points is said to be *convexly independent* if

.

Definition (General position in 3 dimensions)

In 3-space, a set X of points is said to be *convexly independent* if

Examples: A motivating proof in 2 dimensions and a fundamental proof in 3 dimensions

A proof in the planar case

Intuitive Motivation:

How many randomly selected planar points in general position must be chosen to guarantee a convex quadrilateral?

A proof in the planar case

Intuitive Motivation:

How many randomly selected planar points in general position must be chosen to guarantee a convex quadrilateral?
Is it more than 4?

A proof in the planar case

Intuitive Motivation:

How many randomly selected planar points in general position must be chosen to guarantee a convex quadrilateral?
Is it more than 4?
Is it more than 5?

A proof in the planar case

Intuitive Motivation:

How many randomly selected planar points in general position must be chosen to guarantee a convex quadrilateral?
Is it more than 4?
Is it more than 5?

Example (The number of planar points in general position to guarantee a convex quadrilateral)

A proof in the planar case

Intuitive Motivation:

How many randomly selected planar points in general position must be chosen to guarantee a convex quadrilateral?
Is it more than 4?
Is it more than 5?

Example (The number of planar points in general position to guarantee a convex quadrilateral)

See board: A proof that $N_2(4) = 5$.

A proof in the planar case

Intuitive Motivation:

How many randomly selected planar points in general position must be chosen to guarantee a convex quadrilateral?
Is it more than 4?
Is it more than 5?

Example (The number of planar points in general position to guarantee a convex quadrilateral)

See board: A proof that $N_2(4) = 5$.
The proof uses three cases. Note the useful role of the Dirichlet Schubfachprinzip (pigeonhole principle).

A proof in the spatial case

An introductory proof in 3-space: A fundamental proof

How many randomly selected spatial points in general position must be chosen to guarantee a convex 3-simplex in (3-) space?

A proof in the spatial case

An introductory proof in 3-space: A fundamental proof

How many randomly selected spatial points in general position must be chosen to guarantee a convex 3-simplex in (3-) space?

Example ($N_3\,(4)$, the number of spatial points in general position to guarantee a convex 3-simplex)

The proof only requires a direct application of the definition of general position and convex independence.

A proof in the spatial case

An introductory proof in 3-space: A fundamental proof

How many randomly selected spatial points in general position must be chosen to guarantee a convex 3-simplex in (3-) space?

Example ($N_3\,(4)$, the number of spatial points in general position to guarantee a convex 3-simplex)

The proof only requires a direct application of the definition of general position and convex independence. The number

$$N_3\,(4) \;=\; 4$$

Case study: Outline of a partial proof for $N_3(6)$

Hint: The number is 9

Claim:

Consider a set X of nine points in general position in space (3-dimensions). Then the set contains a convex polytope with 6 vertices.

Hint: The number is 9

Claim:

Consider a set X of nine points in general position in space (3-dimensions). Then the set contains a convex polytope with 6 vertices.

Counterexample:

An arbitrary set of eight points does not contain a convex polytope with six vertices. Queensy will demonstrate this crucial step in the proof for you.

Outline of a partial proof that $N_3(6) = 9$

Case: $ConvX \geq 6$

Suppose that the convex hull of a set of points X in general position is ≥ 6. Then the set contains a convex polytope with 6 vertices.

Outline of a partial proof that $N_3(6) = 9$

Case: $ConvX \geq 6$

Suppose that the convex hull of a set of points X in general position is ≥ 6. Then the set contains a convex polytope with 6 vertices.

Case: $ConvX = 5$

Suppose that the convex hull of a set of points X in general position is ≥ 5. We will show in a particular subcase (1 of 2) that the set contains a convex polytope with 6 vertices.

Outline of a partial proof that $N_3(6) = 9$

Case: $ConvX \geq 6$

Suppose that the convex hull of a set of points X in general position is ≥ 6. Then the set contains a convex polytope with 6 vertices.

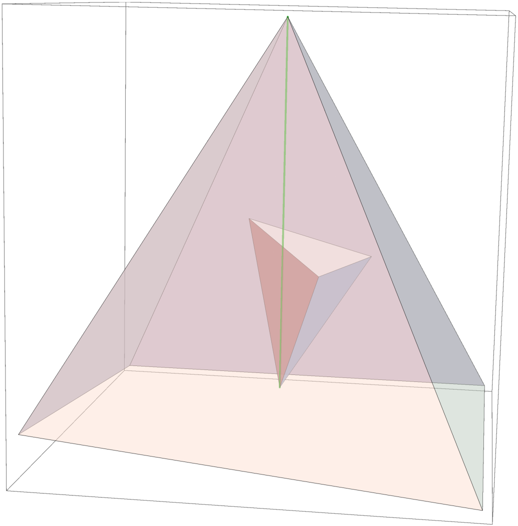
Case: $ConvX = 5$

Suppose that the convex hull of a set of points X in general position is ≥ 5. We will show in a particular subcase (1 of 2) that the set contains a convex polytope with 6 vertices.

Excluded cases

Here, we exclude a subcase of the argument when $ConvX = 5$. We also exclude the case when $ConvX = 4$.

A special case of 9 points with convex hull of size 5



A detail of 9 points with convex hull of size 5