Complex Continued Fractions

Julian Benali, Carlos Verastegui

Mason Experimental Geometry Lab

George Mason University

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Introduction

It is standard to write numbers as a sequence of decimals, such as $\sqrt{2} = 1.41421...$. Another way to write a number is to specify a sequence of descending continued fractions that represent it. In this way, we can write $\sqrt{2} = 1 + \frac{1}{2+\frac{1}{\ddots}}$. This idea can then be extended to complex numbers as well. Continued fractions have lots of great number-theoretic properties. Better yet, keeping track of the numerators and denominators leads to thinking about 2×2 matrices. The goal of this project was to visualize continued fractions and explore the relationship between two natural equivalence relations on continued fractions: tail equivalence and matrix equivalence.

Definition (Continued Fraction)

A *regular continued fraction* is a representation of a real number as a sequence of positive integers in a descending fraction:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \frac{1}{\ddots}}}$$

For brevity, it is standard to write $\pi = [3; 7, 15, 1, 292, ...]$.

Regular Continued Fraction Examples

$$\frac{27}{31} = [0; 1, 6, 1, 3]$$

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, ...]$$

$$\sqrt{2} = [1; 2, 2, 2, 2, ...]$$

$$-\sqrt[3]{10} = [-3; 1, 5, 2, 9, ...]$$

Definition (Complex Continued Fraction)

The idea of continued fractions can then be extended to complex numbers. A *complex continued fraction* is a representation of a complex number of the form:

$$\pi\mathrm{i} = 3\mathrm{i} + \cfrac{1}{-7\mathrm{i} + \cfrac{1}{16\mathrm{i} + \frac{1}{\ddots}}}$$

Complex Continued Fraction Examples

$$\frac{19+16\mathrm{i}}{31} = [1+\mathrm{i}; -1+\mathrm{i}, -4\mathrm{i}, -3-4\mathrm{i}]$$

$$\mathrm{i}\sqrt{2} = [\mathrm{i}, -2\mathrm{i}, 2\mathrm{i}, -2\mathrm{i}, 2\mathrm{i}, ...]$$

$$\sqrt{7} + 3\pi\mathrm{i} = [3+9\mathrm{i}; -1-\mathrm{i}, -1+2\mathrm{i}, 2-4\mathrm{i}, -4-4\mathrm{i}, ...]$$

Gauss Map

The Gauss map is a complex function which shifts the digits of a complex continued fraction. Formally,

$$T(z) = \frac{1}{z} - \left[\frac{1}{z}\right]$$

for any $z \in \{a + b\mathrm{i} : a, b \in [-1/2, 1/2)\}$ where $[x]$ denotes the nearest Gaussian integer function. For example,

$$T([0; -3, 1+4\mathrm{i}, -2]) = [0; 1+4\mathrm{i}, -2]$$

$$T([0; -5, 5, -5, 5, ...]) = [0; 5, -5, 5, -5, ...]$$

Equivalence Relations

Matrix Equivalence

We say that two complex numbers x and y are *matrix equivalent* if there exist Gaussian integers a, b, c, d such that $x = \frac{ay+b}{cy+d}$ and $ad - bc = \pm 1$. If x and y are being represented by 2×2 matrices, then this is equivalent to saying that $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} y$ where $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1$.

Tail Equivalence

We say two continued fractions $x = [a_0; a_1, a_2, ...]$ and $y = [b_0; b_1, b_2, ...]$ are *tail equivalent* if there exist positive integers r and s such that $a_{n+r} = b_{n+s}$ for every positive integer n. In other words, two numbers are tail equivalent if one of their sequences of continued fraction digits can be shifted so that their tails are aligned.

Matrix Equivalence Examples

$$\mathrm{i}\sqrt{2} \sim 3 - 5\mathrm{i} + \mathrm{i}\sqrt{2}$$

$$\frac{1+\sqrt{5}}{2} \sim 1 - \frac{1+\sqrt{5}}{2}$$

$$[0; 3, 4, 5, 6, ...] \sim [0, 4, 5, 6, 7]$$

Tail Equivalence Examples

$$[0; 1, 2, 3, 1, 2, 3, ...] \sim [0; 3, 1, 2, 3, 1, 2, ...]$$

$$\frac{\sqrt[3]{19}}{3} \sim 2 - 5\mathrm{i} + \frac{\sqrt[3]{19}}{3}$$

$$[4; 5\mathrm{i}, 6, 7\mathrm{i}, 8, 9\mathrm{i}, ...] \sim [1; 6, 7\mathrm{i}, 8, 9\mathrm{i}, 10, ...]$$

Illustrations

Figure 1 shows the set of all points $x = [0; a_1, a_2, ...]$ between 0 and 1 whose first regular continued fraction digit a_1 is even. Figure 2 shows the same set of points in blue. In orange is the set of all point $x = [0 : a_1, a_2, ...]$ such that a_1 and a_2 are both even. Figure 3 shows the Gauss map on the window $[-0.5, 0.5] \times [-0.5, 0.5]$ in the complex plane. The dark spots correspond to numbers which map to 0. The tiling illustrates the nearest integer rounding $z - [z]$, and the circular arcs are due to the inversion $1/z$.

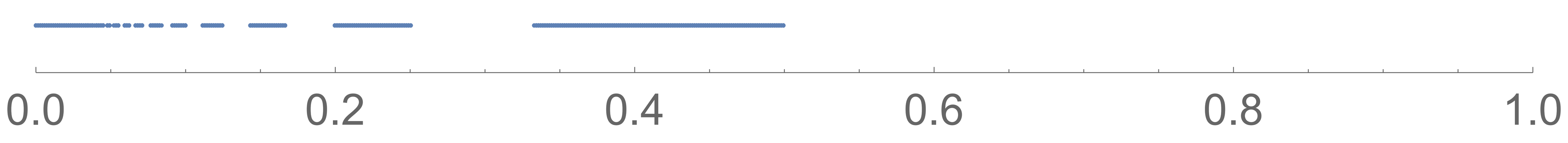

Figure 1

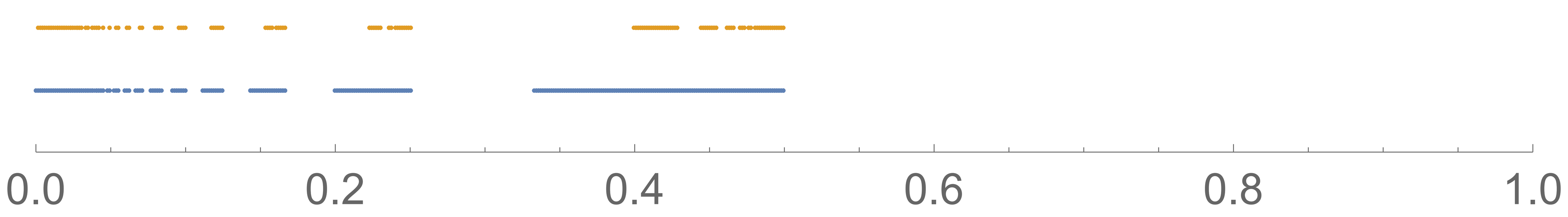

Figure 2

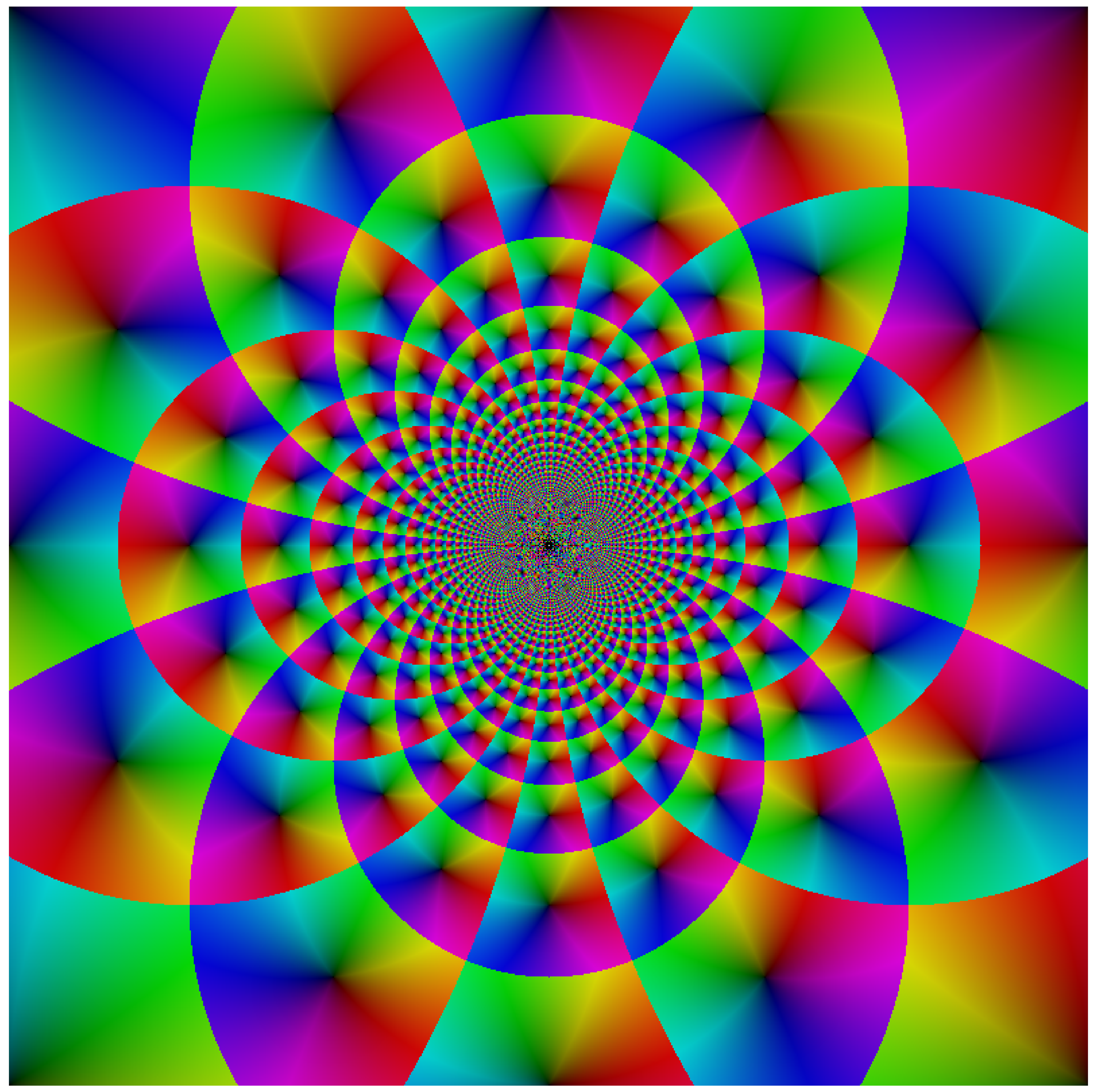

Figure 3

Project Goal

It can be shown that two real numbers are matrix equivalent if and only if they are tail equivalent. For complex numbers, however, tail equivalence implies matrix equivalence, but there exist exceptions to the converse. Previously, the only known example of matrix equivalent numbers which are not tail equivalent was given by Richard Lakein (1974). Thus, it was our goal to find and characterize more of these examples.

Results

- By exploring Lakein's example, we were able to generate new examples of complex numbers which are matrix equivalent, but not tail equivalent.
- We wrote a program which can sieve through many continued fractions and test them for matrix and tail equivalence which we hope can be used to find more examples.
- Here is one of the examples we found:

$$[0; \overline{-2-\mathrm{i}, -2-\mathrm{i}, 3\mathrm{i}, 1-2\mathrm{i}, -1+2\mathrm{i}, -3}, ...] \sim$$
$$[0; \overline{2-\mathrm{i}, -2+\mathrm{i}, -1+2\mathrm{i}, -1-2\mathrm{i}, -1-2\mathrm{i}, 2+\mathrm{i}}, ...]$$