Visualizing Bruhat-Tits Buildings

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Introduction

The aim of the project is to visualize the so called *Bruhat-Tits Building* of the group $GL_3(\mathbb{Q}_2)$, with the intent of generalizing the methods for use with other building types.

Drawing the building is a three step process. Using both the Bruhat decompostion and the root group decomposition of this group, we obtain methods for generating all components (the chambers), deciding which are adjacent (generating the chamber graph) and finally positioning them in space.

Coxeter complexes

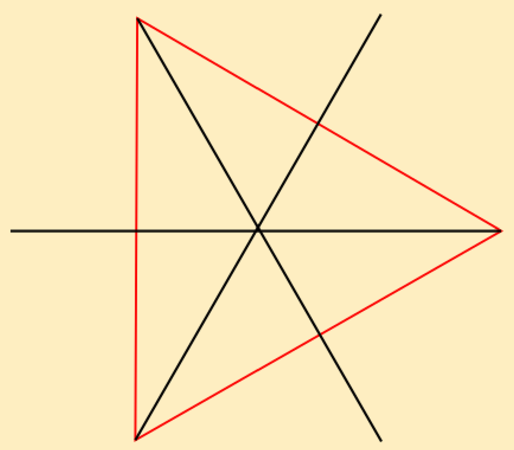

Figure 1: The three reflections generating all linear isometries of the equilateral triangle. This gives the Coxeter complex for S_3.

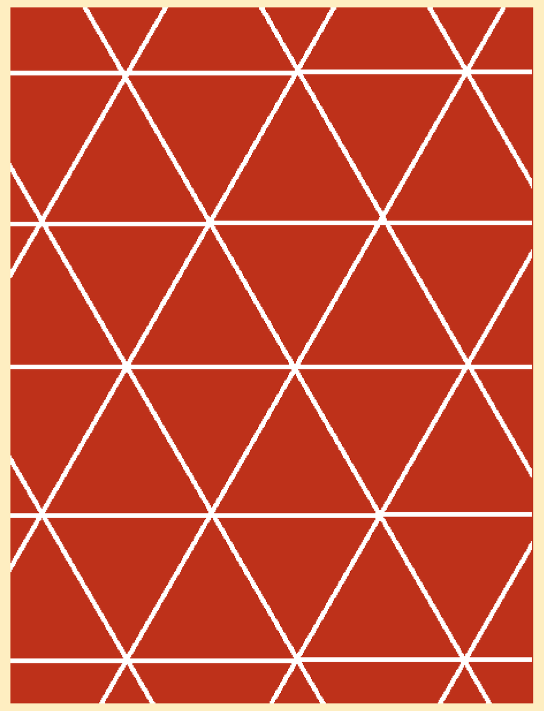

Figure 2: Part of the Coxeter complex of $\mathbb{Z}^2 \rtimes S_3$.

Definition (Coxeter system[1])

A *Coxeter system* is a group W together with a set of generators S, called the set of *simple reflections*, such that W allows a presentation:

$$W = \langle S \mid \forall i, j : s_i^2 = (s_i s_j)^{m(i,j)} = e \rangle.$$

Definition (Coxeter complex[1])

For a Coxeter system (W, S) with $\#S = n$, the Coxeter complex is the hyperplane arrangement in R^n generated by the simple reflections, with angles between two hyperplanes given by $\pi / m(i,j)$, with $m(i,j)$ the order of $s_i s_j$.

The areas enclosed by hyperplanes are called the *chambers* of the complex.

The Coxeter group associated to $GL_3(\mathbb{Q}_2)$ is $W := \mathbb{Z}^2 \rtimes S_3$. S_3 is the group of linear isometries of the triangle (figure 1). W then is the group of all isometries, giving a tessellation of the plane by equilateral triangles (figure 2). It is generated by three simple reflections.[1]

Visualizing buildings

Definition (Building[1])

A building is the union of Coxeter complexes called *apartments*. For any pair of chambers A, B there is an apartment containing both. For any two apartments that both contain A and B, there is an isomorphism between the two apartments.

In particular, all apartments are isomorphic Coxeter complexes.

Definition (Strong Transitivity[1])

A group is said to act strongly transitively on a building if it acts transitively on the chambers, and the stabilizer of any chamber acts transitively on the apartments.

If G acts strongly transitively on the building, the chambers correspond one-to-one to cosets of a subgroup $B \subset G$, which consists of matrices that are upper triangular modulo $T(2\mathcal{O}_{\mathbb{Q}_2})$. The main component of our visualization methods is the Bruhat decomposition $G = \coprod_{w \in W} BwB$.

Computational aspects

Generateing the chamber labels

By the Bruhat decomposition we can find representatives for the cosets of B by $BwB / (BwB \cap B) \cong B / (wBw^{-1} \cap B)$. There exists a bijection between such a quotient and a product of finite cyclic groups of order 2^q for certain non-negative q. [2, 3]

Generate the chamber graph

We can reconstruct the chamber graph from the fact that two chambers gB and hB are adjacent for simple reflection $s \in S$ if and only if $Bg^{-1}hB = BsB$. The form of elements of BsB can easily be calculated.[1]

Determine the positions of all vertices

The precise position of a chamber in space gB is determined by the double coset BwB it is contained in, which in turn only depends on w. Each chamber is an equilateral triangle whose orientation also only depends on w.

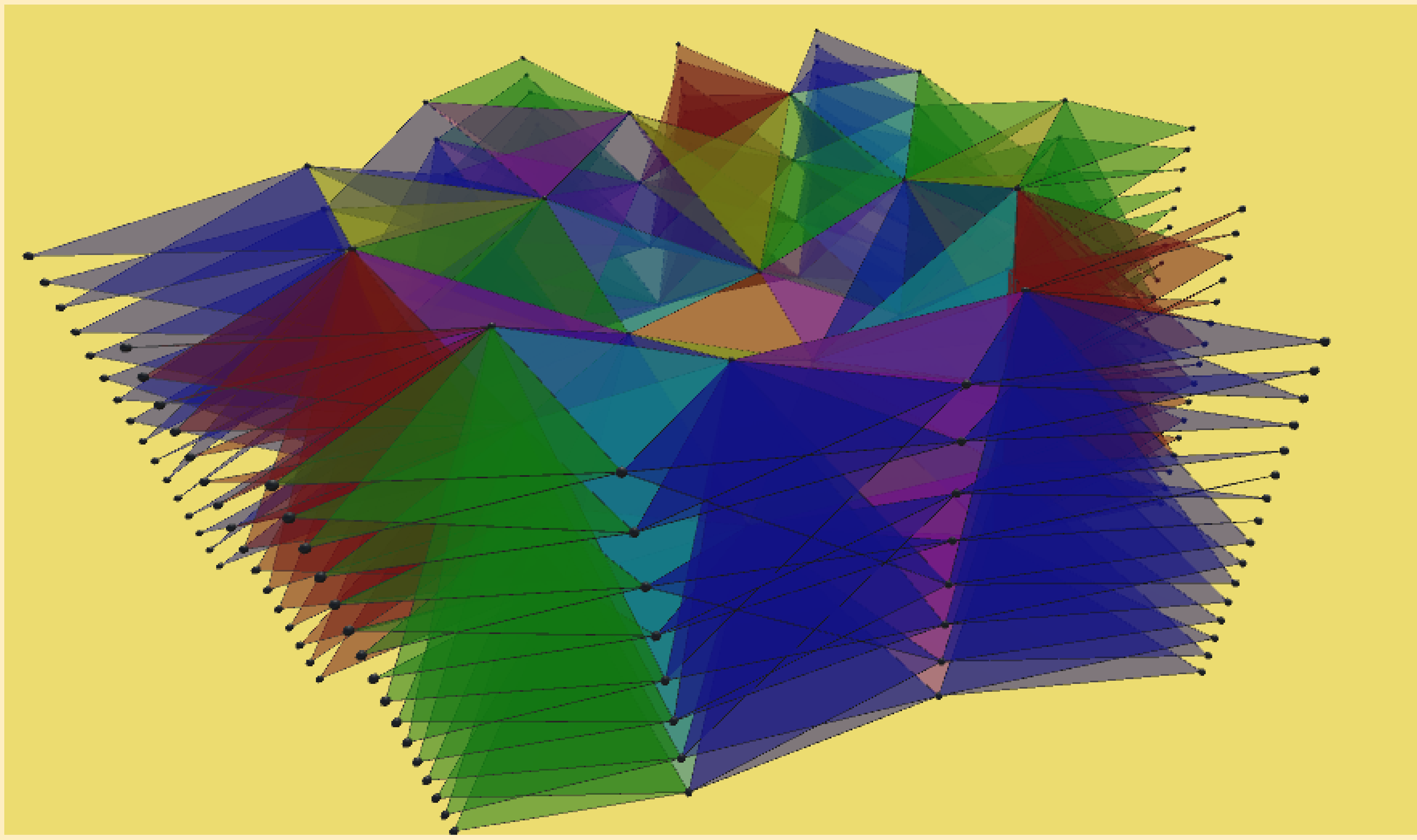

Figure 3: The Bruhat-Tits building of $GL_3(\mathbb{Q}_2)$ up to distance 4 from the fundamental chamber

Conclusions/Future Work

This project consists of mostly preliminary research in the topic. Based on the results, it is safe to say that there is a world of possibilities in visualizing buildings, and certainly it will lead to beautiful imagery useful for both outreach and education.

To increase the usefulness, methods for visualizing the different apartments and animating retractions can be developed. Another useful addition would be to add the functionality to draw galleries in the building, and animate where retractions take them.

The most obvious routes to take this research include optimizing the code for time efficiency and exploring the implications of changing the type of the building. For example, even affine type B_2 might already have its own difficulties.

Aside of this, a few basic methods are left unexplored. For example, the building discussed here can also be realized as the incidence geometry associated to the $\mathcal{O}_{\mathbb{Q}_2}$-lattices in $(\mathbb{Q}_2)^3$. This raises the question whether computing the chamber graph from this data would be more efficient.

Acknowledgments

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