Becoming a Fan of the Gröbner Fan: Combinatorial Algebra of a Modulized Basis

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MEGL

May 6th, 2016

Goal

To develop tools to understand a compactification of the variety $\chi(F_3, SL_2)$

We refer to this compactification as $M \subseteq \mathbb{P}^7$ where $\mathbb{P}$ is projective space

Definitions:

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- Polytope:
  Finite intersection of halfspaces in some $\mathbb{R}^d$ that is bounded, meaning it does not contain a ray $\{\mathbf{x} + t\mathbf{y} : t \geq 0\}$ for any $\mathbf{y} \neq \mathbf{0} \rightarrow$ H-Polytope

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  equivalently

  Convex hull of a finite set of points in some $\mathbb{R}^d \rightarrow$ V-Polytope

- Polytope Face:
  A *face* of a polytope $P \in \mathbb{R}^n$ is defined

  $$F_w = \{\mathbf{u} \in P : \mathbf{w} \cdot \mathbf{u} \geq \mathbf{w} \cdot \mathbf{v} \forall \mathbf{v} \in P\}$$

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  where $\mathbf{w} \in \mathbb{R}^n$ is called a *normal vector* to the face F_w where $\mathbf{w}$ is not necessarily unique
- Dimension of P:
  $dim(P) = \#$ of $\mathbf{w_i}$ for face $F + dim(F)$

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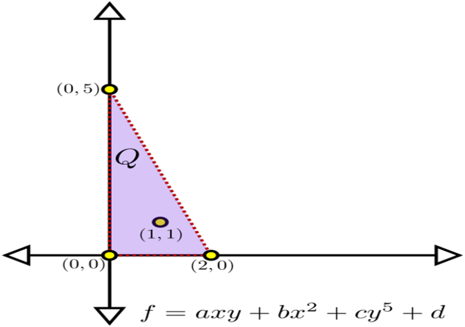
Ex:



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$$< in_w(f) : f \in I > .$$

  - If we have a principal polynomial ideal (ideal generated by one polynomial $I =< f >$), then

$$in_w(I) =< in_w(f) > .$$

Ex:

$$w = (1, 1, 0)$$

$$f(x, y, z) = xy + xz + x^2$$

$$in_w(f) = xy + x^2$$

- Cone:
  A *cone* C is a polyhedron in $\mathbb{R}^n$ such that for all $u, v \in C$, $a \in \mathbb{R}^+$,
  1. $u + v \in C$
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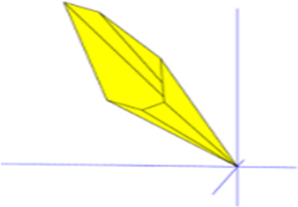
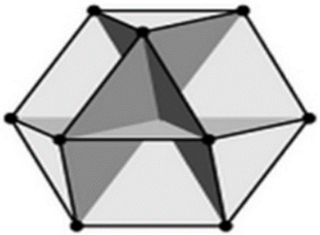
- Polyhedral Cone:
  A cone with finitely many generators $\{u_1, \ldots u_k\} \subset C$ i.e. $C = \{\lambda_1 u_1 + \ldots + \lambda_k u_k : \lambda_i \geq 0\}$.



Polyhedral Fan:
Finite collection of polyhedral cones such that the intersection of any finite number of cones is a cone in the fan.



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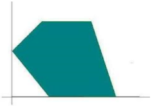
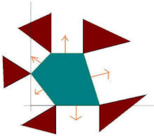
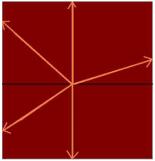

- Gröbner Fan:
  For an ideal I in $K[X]$ we define the *Gröbner fan* $GF(I)$ to be the set of closed cones $C[w]$ where

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- We will see shortly that the Gröbner fan is useful in our research

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Let I be a homogeneous ideal in $K[X]$. There exists a polytope $State(I) \subset K^n$ whose normal fan coincides with the Gröbner fan $GF(I)$.

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Fact 2:
Let I be a principle homogeneous ideal $I =< f >\in K[X]$, then $State(I) = New(f)$.

Fact 3:
The edges of the state polytope of an ideal I are in a natural bijection with the distinct binomial initial ideals $in_w(I)$.

Polynomial from "Compactifications of Character Varieties and Skein Relations on Conformal Blocks" by Christoper Manon:

$$x_{111}x_{100}x_{010}x_{001} - x_{110}x_{101}x_{011}x_{000}$$
$$+x_{110}^2x_{000}^2 + x_{101}^2x_{000}^2 + x_{011}^2x_{000}^2$$
$$+x_{100}^2x_{000}^2 + x_{010}^2x_{000}^2 + x_{001}^2x_{000}^2 + x_{111}^2x_{000}^2 - 4x_{000}^4$$
$$+x_{111}x_{100}x_{011}x_{000} + x_{111}x_{010}x_{101}x_{000} + x_{111}x_{001}x_{110}x_{000}$$
$$+x_{100}x_{010}x_{110}x_{000} + x_{100}x_{001}x_{101}x_{000} + x_{010}x_{001}x_{011}x_{000} = 0$$

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- Polynomial gives neat hypersurface
- Solution set to this polynomial gives variety

Using trivalent graphs and these edge weights, Dr. Manon has conjectured that

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is an irreducible edge of the polytope.

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We hope to prove this

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    311 dimension 2 faces

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Which of these are edges of the polytope?

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Algorithm for finding polytope edges developed by Dr. Manon:

1. Using dim 7 cones, sum corresponding extremal rays $\rightarrow W$
2. Compute initial form of W
   2.1 Find monomials whose exponent vectors are weighted highest by W
3. This is edge - check against irreducible binomials

W and Its Initial Form

```
fullwvec1 = []
for vec1 in dim7cones:
    wvec1 = [0, 0, 0, 0, 0, 0, 0, 0]
    for num1 in vec1:
        rel_vec1 = rays[num1]
        wvec1 = [h + k for h, k in zip(wvec1, rel_vec1)]
    fullwvec1.append(wvec1)
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Ex: dim 7 cone $= (0, 1, 2, 3, 4, 5, 6)$

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Ex: dim 7 cone $= (0, 1, 2, 3, 4, 5, 6)$

Add rays $0 + 1 + 2 + 3 + 4 + 5 + 6 \rightarrow [-13, -7, 3, 3, -3, 7, 5, 5]$

W and Its Initial Form

```
list = []
for vec2 in fullwvec1:
    listv2 = []
    a1 = vec2[0] + vec2[1] + vec2[2] + vec2[4]
    a2 = vec2[5] + vec2[6] + vec2[7] + vec2[3]
    a3 = vec2[5]*2 + vec2[3]*2
    a4 = vec2[6]*2 + vec2[3]*2
    a5 = vec2[7]*2 + vec2[3]*2
    a6 = vec2[1]*2 + vec2[3]*2
    a7 = vec2[2]*2 + vec2[3]*2
    a8 = vec2[4]*2 + vec2[3]*2
    a9 = vec2[0]*2 + vec2[3]*2
    a10 = vec2[3]*4
    a11 = vec2[0] + vec2[1] + vec2[7] + vec2[3]
    a12 = vec2[0] + vec2[2] + vec2[6] + vec2[3]
    a13 = vec2[0] + vec2[4] + vec2[5] + vec2[3]
    a14 = vec2[1] + vec2[2] + vec2[5] + vec2[3]
    a15 = vec2[0] + vec2[4] + vec2[6] + vec2[3]
    a16 = vec2[2] + vec2[4] + vec2[7] + vec2[3]
    listv2 = [a1, a2, a3, a4, a5, a6, a7, a8, a9,
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    listv2 = [a1, a2, a3, a4, a5, a6, a7, a8, a9,
    list.append(listv2)
```

Take the monomials in f whose exponent vectors are weighted highest by W

Ex: -13(1) -7(1) + 3(1) + 3(0) -3(1) + 7(0) + 5(0) + 5(0) = -20

W and Its Initial Form

```
indlist = []
func = [" + xyza", " - bcdw", " + b^2w^2", " + c^2w^

for vec in list:
    m = max(vec)
    ind = [i for i, j in enumerate(vec) if j == m]
    indlist.append(ind)
print indlist

for vec2 in indlist:
    ans = " "
    if len(vec2) == 2:
        for num in vec2:
            a = func[num]
            ans = ans + a
        print(ans)
```

Add the monomials these map to

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Add the monomials these map to

If one of the irreducible binomials is this polynomial, it's an edge

Results

$$x_{110}^2x_{000}^2 - 4x_{000}^4 \tag{1}$$

$$x_{100}^2x_{000}^2 - 4x_{000}^4 \tag{2}$$

$$-x_{110}x_{101}x_{011}x_{000} + x_{011}^2x_{000}^2 \tag{3}$$

$$x_{101}^2x_{000}^2 + x_{011}^2x_{000}^2 \tag{4}$$

$$x_{110}^2x_{000}^2 + x_{011}^2x_{000}^2 \tag{5}$$

$$x_{111}x_{100}x_{010}x_{001} + x_{010}x_{001}x_{011}x_{000} \tag{6}$$

$$x_{011}^2x_{000}^2 - 4x_{000}^4 \tag{7}$$

Results

$$x_{010}^2 x_{000}^2 - 4x_{000}^4 \tag{8}$$

$$x_{101}^2 x_{000}^2 - 4x_{000}^4 \tag{9}$$

$$-x_{110}x_{101}x_{011}x_{000} + x_{110}^2 x_{000}^2 \tag{10}$$

$$-x_{110}x_{101}x_{011}x_{000} + x_{101}^2 x_{000}^2 \tag{11}$$

$$x_{101}^2 x_{000}^2 + x_{100}^2 x_{000}^2 \tag{12}$$

$$x_{001}^2 x_{000}^2 - 4x_{000}^4 \tag{13}$$

$$x_{110}^2 x_{000}^2 + x_{101}^2 x_{000}^2 \tag{14}$$

$$x_{111}x_{100}x_{010}x_{001} + x_{111}x_{010}x_{101}x_{000} \tag{15}$$

$$x_{111}x_{100}x_{010}x_{001} + x_{100}^2 x_{000}^2 \tag{16}$$

Results

$$x_{111}^2 x_{000}^2 - 4x_{000}^4 \tag{17}$$

$$x_{011}^2 x_{000}^2 + x_{111}^2 x_{000}^2 \tag{18}$$

$$x_{110}^2 x_{000}^2 + x_{111} x_{001} x_{110} x_{000} \tag{19}$$

$$x_{111}^2 x_{000}^2 + x_{111} x_{001} x_{110} x_{000} \tag{20}$$

$$x_{101}^2 x_{000}^2 + x_{010}^2 x_{000}^2 \tag{21}$$

$$x_{010}^2 x_{000}^2 + x_{010} x_{001} x_{011} x_{000} \tag{22}$$

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Code currently shows that none of the irreducible binomials are edges

We're working on that

- There exists a weighting that gives us the initial ideal $(4, 4, 4, 3, 3, 3, 3, 0)$ that gives

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- Then the modulized space M (a compactification of $\chi(F_3, SL_2)$) has a Newton Okounkov Body
    - A convex body in Euclidean space associated to a divisor on a variety
    - A far generalization of the Newton polytope of a projective toric variety

Acknowledgements

Dr. Chris Manon
Dr. Sean Lawton
MEGL
National Science Foundation